DCPT-08/63

Black Rings in Taub-NUT and D0-D6 interactions

Joan Camps[a], Roberto Emparan[a,b], Pau Figueras[c], Stefano Giusto[d], Ashish Saxena[d]

[a] *Departament de Física Fonamental, Universitat de Barcelona, Martí i Franquès 1, E-08028, Barcelona, Spain*

[b] *Institució Catalana de Recerca i Estudis Avançats (ICREA) Passeig Lluís Companys, 23, E-08010 Barcelona, Spain*

[c] *Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, South Road, Durham, DH1 3LE, UK*

[d] *Institut de Physique Théorique, CEA/Saclay 91191 Gif sur Yvette, France*

`jcamps@ffn.ub.es`, `emparan@ub.edu`, `pau.figueras@durham.ac.uk`, `stefano.giusto@cea.fr`, `ashish.saxena@cea.fr`

Abstract

We analyze the dynamics of neutral black rings in Taub-NUT spaces and their relation to systems of D0 and D6 branes in the supergravity approximation. We employ several recent techniques, both perturbative and exact, to construct solutions in which thermal excitations of the D0-branes can be turned on or off, and the D6-brane can have B-fluxes turned on or off in its worldvolume. By explicit calculation of the interaction energy between the D0 and D6 branes, we can study equilibrium configurations and their stability. We find that although D0 and D6 branes (in the absence of B fields, and at zero temperature) repeal each other at non-zero separation, as they get together they go over continuosly to an unstable bound state of an extremal singular Kaluza-Klein black hole. We also find that, for B-fields larger than a critical value, or sufficiently large thermal excitation, the D0 and D6 branes form stable bound states. The bound states with thermally excited D0 branes are black rings in Taub-NUT, and we provide an analysis of their phase diagram.

Contents

1 Introduction and Summary

Recently there has been great progress in advancing techniques to construct and analyze solutions for higher-dimensional black holes [1, 2, 3]. These black holes allow for non-spherical topologies as well as extended horizons, and can often be related to self-gravitating D-brane configurations. The progress has mainly come from two different lines: (i) five-dimensional vacuum solution-generating techniques have yielded many qualitatively new solutions describing black rings and black holes [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]; (ii) approximate methods have allowed to construct and analyze thin black rings in a larger variety of backgrounds and dimensions [14, 15].

In this paper we bring to bear, and at some points refine and extend, these techniques to analyze black rings in backgrounds of Taub-NUT type. Such solutions describe, when embedded in M-theory and then reduced to IIA theory, D0-branes in the presence of D6-branes. The self-gravitating D6-brane is essentially a Kaluza-Klein (KK) monopole plus six additional space dimensions of M-theory, while the D0 brane uplifts to a momentum wave of gravitons along the eleventh direction. When the D0 is excited thermally, this graviton wave develops a horizon and becomes a boosted black string — conversely, when the boost becomes light-like and the horizon shrinks to zero, we recover the gravitational description of a D0 brane in its supersymmetric ground state. In the Taub-NUT background, where the eleven-dimensional direction is contractible, the black string is actually a black ring. So, quite generally, arrangements of D0 and D6 branes are described in supergravity as black rings in Taub-NUT. As four-dimensional solutions, they describe a (singular) magnetic monopole and an electrically charged black hole. The KK magnetic and electric charges, P and Q, are related to the numbers N_6 and N_0 of D6 and D0 branes through the length of the compact Kaluza-Klein circles near asymptotic infinity, $2\pi L$,

$$P = \frac{LN_6}{4}\,, \qquad Q = \frac{2G_4 N_0}{L}\,. \tag{1.1}$$

For the most part we will consider for simplicity a single D6 brane, $N_6 = 1$. Configurations of this sort have been constructed in the supersymmetric case in [16, 17, 18]. Our aim is to study the much more complex non-supersymmetric D0-D6 systems.

The D0-D6 system presents a number of peculiar features. The long-distance D0-D6 interaction, mediated by NSNS gravi-dilaton and RR gauge-field closed-string exchange, is repulsive. This admits a simple interpretation in M-theory, where the effect is simply the centrifugal force created by the rotation of a ring with light-like local boost. It must be noted, though, that despite this long-distance repulsion, D0 branes can bind to the worldvolume of D6 branes to form non-supersymmetric metastable bound states [19] which, at strong coupling, can be precisely matched to Kaluza-Klein black holes with non-zero Bekenstein-Hawking entropy [20, 21, 22, 23].

It is known that the physics of D0-D6 interactions becomes richer when B-form fluxes, introduced as moduli, are turned on in the worldvolume of the D6 [24, 25, 26]. When the fluxes are large enough (more precisely, when a codimension-1 wall is crossed in the moduli space of B-fields) it is possible to have supersymmetric bound states of D0 and D6 [24, 26], where the D0 is at a finite distance from the D6. But we can envisage another way of achieving equilibrium between a set of D0 branes and a D6 brane (without B-fields). If we add some energy of excitation to the D0 branes while keeping their charge fixed, we enhance the gravitational attraction to the D6 brane, which may then overcome their repulsion. When the excitations of the D0 branes (*i.e.,* of the open strings stretched between a gas of D0s) have a thermal distribution, then in the regime of validity of supergravity they are described as D0-charged

black holes. Thus, if the horizon area of the D0-charged black hole is large enough, a non-supersymmetric bound state may be possible. A main aim of this paper is to demonstrate these two mechanisms using the novel gravity techniques mentioned above.

Finding equilibrium configurations is not the only information we can obtain from our methods: we can also study their stability and their interaction energy. In constrast to the techniques based on solving Killing spinor equations, we can construct configurations in which the separation R between D0 and D6 branes does not correspond to equilibrium (so supersymmetry is broken). We compute the interaction energy as the difference between the total ADM energy of the system, as measured at infinity, and the masses of the D0 and D6 branes when they are isolated from each other,

$$E_{\rm int}(R) = M_{\rm tot}(R) - M_{\rm D0} - M_{\rm D6} \,. \tag{1.2}$$

The D0-D6 separation in equilibrium states corresponds to extrema of this energy for fixed charges and horizon area, essentially as a consequence of the first law. *Stable* configurations should correspond to minima. When the D0s are not excited and so have zero entropy, $M_{\rm D0}$ is simply determined by its charge (*i.e.*, net number of D0 branes). When the D0s are thermally excited, we take $M_{\rm D0}$ to be the mass of a D0-charged black hole with fixed values of the charge and area (entropy). Thus $E_{\rm int}$ measures the interaction energy as the thermally-excited D0 branes are moved adiabatically towards the D6 brane[1]. For the perturbative solutions a convenient alternative way to determine the stability is to analyze the external force needed to balance the configurations away from equilibrium— a potential associated to this force can also be constructed, which is closely related to $E_{\rm int}$.

Our main results are:

1. In the absence of B-fields, the closed-string interaction between the D0 and D6 at any finite separation is repulsive. However, the interaction energy goes continuously to a finite maximum as the distance between the D0 and the D6 decreases to zero. We construct a family of exact solutions that in this limit describe the formation of an unstable D0-D6 bound state corresponding to an extremal (singular) Kaluza-Klein black hole with angular momentum $J = PQ/G_4$ and mass

$$M_{\rm bh} = E_{\rm int} + M_{\rm D0} + M_{\rm D6} \,. \tag{1.3}$$

   See figure 1. To obtain this result it is crucial to work with the exact solutions: perturbative calculations break down as the D0 and D6 get together, since they give $E_{\rm int} \to \infty$.

2. When B-fields are turned on in the D6 worldvolume, the interaction is again repulsive below a critical value B_c, but for $B \geq B_c = L/2\sqrt{3}$ a stable minimum develops. The equilibrium solutions we find, using an approximate construction of thin black rings, reproduce precisely previous results based on rather different, supersymmetry-based, techniques. Our methods also provide the off-shell interaction energy and thus a simple way to check the stability of these configurations. See figure 2.

3. Thermally excited D0-branes, even in the absence of B-fluxes, can achieve equilibrium configurations in the presence of D6 branes if the entropy of the D0-brane thermal gas is

[1] Alternatively, we could fix the energy of the D0s, in which case the configuration of stable equilibrium would be that which maximizes the entropy. The results are in both cases essentially the same, and for definitiness we choose the fix the area in order to determine $M_{\rm D0}$.

larger than a critical value

$$S > S_c = 8\sqrt{2}\pi N_0^2 \frac{G_4}{L^2}\,, \tag{1.4}$$

or equivalently, if the mass of the excited D0s is

$$M > M_c = \frac{3}{\sqrt{2}}\frac{N_0}{L}\,. \tag{1.5}$$

These are proper black rings in Taub-NUT space, in the sense that they have regular horizons of finite area. Again, we construct both exact and perturbative solutions for such systems. For the perturbative solutions we obtain $E_{\rm int}$, which allows to argue the stability of the bound states under changes in the distance between the D0 and D6, see figure 3. The exact solutions that we construct are only a subfamily of the most general class of exact solutions for black rings in Taub-NUT, since we cannot vary independently the S^1 and S^2 angular momenta of the black ring.

Point 1 above deserves further comment. As we have mentioned, ref. [19] described how D0 branes can bind to the worldvolume of D6 branes and form quadratically (meta-)stable bound states. The construction in [19] did not include any angular momentum. However, one expects that angular momentum can be added in the form of fermionic excitations of the 0-6 open strings. When these fill up to the Fermi level, the configuration will have angular momentum $J = N_0 N_6/2$, and vanishing macroscopic entropy. This is precisely like in the extremal Kaluza-Klein black hole with $J = PQ/G_4$. Our result (1.3) amounts to an *exact* computation of the mass of this bound state by taking into account the energy of closed-string interaction stored in the bound state as N_0 D0 branes are moved towards N_6 D6 branes.[2] Moreover, the fact that the exact interaction energy reaches a maximum that accounts precisely for the mass of the black hole strongly suggests that, for this limiting value of the angular momentum, the state of the D0 branes in the worldvolume of the D6 brane with $J = N_0 N_6/2$ is not a metastable minimum as the one with $J = 0$ in [19][3] but is actually an unstable maximum. It would be interesting to derive this result from an analysis of the D6 worldvolume gauge theory.

Regarding point 3, we remark that the entropy and mass of the thermal D0s will not only be bounded below for given N_0, but they will also be bounded *above*. This is, if the D0 branes become too massive, the repulsive effect between D0 and D6 charges will be overwhelmed by the attraction between their masses. In terms of black rings, this corresponds to the fact that the mass and area of a black ring with a given spin along the S^1 are bounded above, the upper values corresponding to the solutions where the thin and fat branches of black rings meet. This regime is away from our perturbative techniques, and we cannot obtain the precise dependence of these upper bounds on N_0. For very small D0 charge, however, the values can be approximately obtained from the asymptotically flat case,

$$S \leq \frac{4\pi}{3\sqrt{3}} N_0\,, \qquad M \leq \left(\frac{N_0}{\sqrt{2G_4 L}}\right)^{2/3} \qquad (G_4 M \ll L)\,. \tag{1.6}$$

The paper is structured as follows: section 2 develops the general technique for approximate perturbative solutions of thin black rings. Here we follow and expand on [14, 15]. In section 3

[2]Observe that given our definition (1.2) and that $M_{\rm bh}$ is also an ADM mass, what makes this result non-trivial is that the black hole solution can be reached continuously from the solutions for separate D0 and D6 branes.

[3]Whose local potential is created by open string interactions.

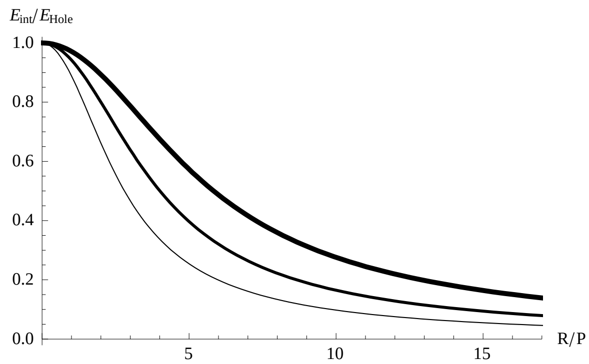

Figure 1: D0-D6 interaction energy $E_{\rm int}$ (with no B-field and at zero temperature) as a function of separation R, for $Q/P = .1$ (thin), 1. (thick) and 10 (thicker). The interaction energy is normalized relative to the one of the (singular) extremal black hole with the same electric and magnetic charges, and angular momentum $G_4 J = PQ$. Here $E_{\rm int}$ is computed using the exact solutions of sec. 5.

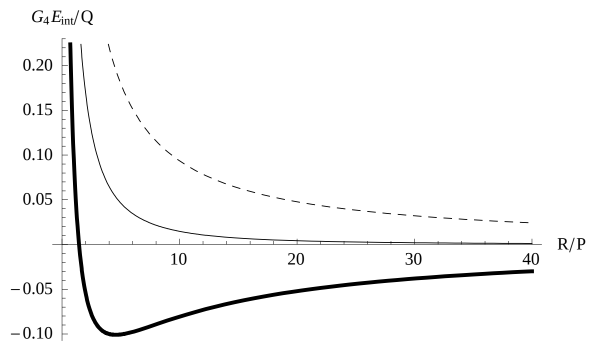

Figure 2: D0-D6 interaction potential with flux $B = bL/2$ versus R, for $b = 1/9$ (dashed), $b_c = 1/\sqrt{3}$ (thin solid) and .8 (thick solid). The interaction energy is normalized with the electric charge so it corresponds to a fixed net number of D0 branes. The potential is obtained from the perturbative extremal solutions of sec. 3.

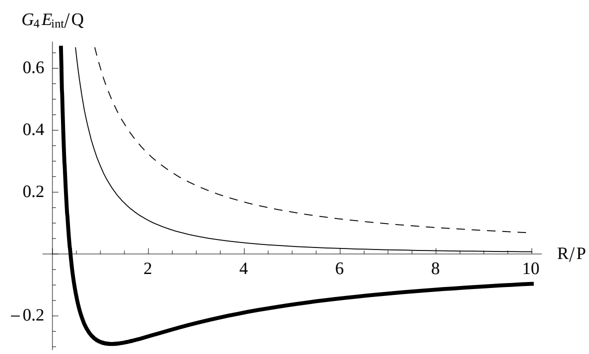

Figure 3: Thermal D0-D6 interaction potential versus R, for fixed values of the entropy and charge, $G_4 S/(4\pi Q^2) = .1$ (dashed), critical $1/\sqrt{2}$ (thin solid) and 3. (thick solid). The interaction energy is normalized with the electric charge. The potential is obtained from the perturbative non-extremal solutions of sec. 3 with $b = 0$.

this technique is applied to put D0 branes in the background of a D6 brane, possibly with B fluxes on the D6 and finite temperature on the D0s. This yields the equilibrium configurations discussed in points 2 and 3 above and the results for the interaction energy plotted in figs. 2

and 3. Section 4 describes the exact solution-generating method (following [10, 12]) and then proceeds to generate the basic solution for a black ring in Taub-NUT. In section 5 we particularize to extremal D0 branes, and compute the interaction energy presented in fig. 1. We also exhibit the limit, discussed in point 1 above, to an extremal KK black hole as the separation between the D0 and D6 vanishes. Section 6 studies a family of exact black rings in Taub-NUT, and discusses their phase diagram. We conclude in section 7.

Note added: H. Yavartanoo has informed us of the paper [27], which deals with related issues.

2 Perturbative approach to thin rings: General method

The first approach is based on the method developed in [14] for a systematic perturbative construction of thin black rings in a background that possesses a $U(1)$ isometry. The ring lies along an orbit of the isometry, and at the zero-th order level of approximation that we work on in this paper, its backreaction on the geometry will be neglected.

The method requires the existence of two widely separated scales, one of which is the ring's S^2-thickness, r_0, and the other one is a large scale $R \gg r_0$ that is typically either a measure of the ring's radius or a characteristic length scale of the background — whichever of the two is smaller. The method allows to determine readily the range of validity of the approximation. At scales much larger than r_0, we can obtain the linearized field created by the ring by substituting it with a distributional source of energy-momentum $T_{\mu\nu}$.

Ref. [28] showed that when a brane with distributional energy momentum $T^{\mu\nu}$ and worldvolume spanning a submanifold of extrinsic curvature tensor $K_{\mu\nu}{}^{\sigma}$, is subject to an external force density $\mathcal{F}^{\sigma}$ along a direction transverse to its worldvolume, then it must satisfy the equations of motion

$$\mathcal{F}^{\sigma} = T^{\mu\nu} K_{\mu\nu}{}^{\sigma} \,. \tag{2.1}$$

In the absence of external forces, $\mathcal{F}^{\sigma} = 0$, this equation imposes a constraint on the sources one can place in a given curved submanifold. In the case of a ring, the circle where the ring lies typically has non-zero extrinsinc curvature, so the equation

$$T^{\mu\nu} K_{\mu\nu}{}^{\sigma} = 0 \tag{2.2}$$

determines the value of the rotation (locally a boost) for which the centripetal ring tension and the centrifugal repulsion balance each other — recall that the gravitational self-interaction of the ring is neglected in this approximation.

By analyzing how this force changes as we change the ring radius, we obtain information about the radial stability of a ring. If to increase the ring radius we need to apply an outward-pushing force, then the ring will be radially stable. If instead we have to push inward to keep the ring in a sligthly larger radius, the equilibrium will be unstable [29].[4] The same information can be put in a perhaps more convenient way if we first integrate the force (2.1) along the ring's radial direction, to obtain a potential for the ring in this background. The two situations described above then correspond to minima and maxima of this potential.

[4] Bear in mind that the force depends on the radius through the geometry, $K_{\mu\nu}{}^{\sigma}$, but also possibly through the ring's parameters in $T_{\mu\nu}$, since typically we will want to keep a physical parameter (such as mass, charge, or area) fixed as the radius is varied.

2.1 Local analysis

We take the isometry of the background along which the ring lies as being parametrized by a coordinate z. The location of the ring can be conveniently specified as the zero of some coordinate ρ measuring radial distance transverse to the circle. Then, close to $\rho = 0$ we can always write the background geometry, to lowest order in ρ/R, as flat space in the form

$$ds^2 = -d\tau^2 + dz^2 + d\rho^2 + \rho^2(d\vartheta^2 + \sin^2\vartheta d\phi^2) + O(\rho/R)\,. \tag{2.3}$$

In this background, we place a distributional source of energy-momentum that reproduces the asymptotic field created by a black *string* in a flat space background. For a boosted black string, this is

$$\begin{aligned} T_{\tau\tau} &= \frac{r_0}{16\pi G_5}\left(\cosh^2\alpha + 1\right)\ \delta^{(3)}(\rho)\,, \\ T_{\tau z} &= \frac{r_0}{16\pi G_5}\ \cosh\alpha\sinh\alpha\ \delta^{(3)}(\rho)\,, \\ T_{zz} &= \frac{r_0}{16\pi G_5}\left(\sinh^2\alpha - 1\right)\ \delta^{(3)}(\rho)\,. \end{aligned} \tag{2.4}$$

One case of particular interest is the extremal limit in which the boost becomes light-like, $\alpha \to \infty$ with $r_0\cosh^2\alpha = p$ finite,

$$T_{\tau\tau} = T_{\tau z} = T_{zz} = \frac{p}{16\pi G_5}\ \delta^{(3)}(\rho)\,. \tag{2.5}$$

Since $T_{\tau z}$ gives the momentum carried by the string along z, and we assume that z is a periodic coordinate, $z \sim z + \Delta z$, in a quantum theory the parameters will be quantized,

$$\frac{r_0}{8\pi G_5}\cosh\alpha\sinh\alpha(\Delta z)^2 = N_0\,, \tag{2.6}$$

with integer N_0. In the context of this paper, in which the direction along the string is dimensionally reduced to obtain IIA solutions, the integer N_0 typically corresponds to the net number of D0 branes, and possibly a contribution to the (quantized) four-dimensional angular momentum.

As mentioned above, the ring must satisfy the equations of motion (2.1). Since these are local equations at the position of the ring, we can analyze them most easily and most generally by considering the spacetime geometry close to the ring. The extrinsic curvature of the ring's circle is a $O(1/R)$ effect, and thus to account for it we must go beyond the zero-th order background (2.3) and include corrections to first order in ρ/R. A wide class of backgrounds are covered by considering corrections of the form

$$\begin{aligned} ds^2 \;=\; & -\left(1 + C_{\tau\tau}\frac{2\rho\cos\vartheta}{R}\right)d\tau^2 + \left(1 + C_{zz}\frac{2\rho\cos\vartheta}{R}\right)dz^2 + 2C_{\tau z}\frac{2\rho\cos\vartheta}{R}d\tau dz \\ & + 2C_{\tau\phi}\frac{2\rho\sin\vartheta}{R}\rho\sin\vartheta d\phi d\tau + 2C_{z\phi}\frac{2\rho\sin\vartheta}{R}\rho\sin\vartheta d\phi dz \\ & + \left(1 + C_{\rho\rho}\frac{2\rho\cos\vartheta}{R}\right)\left(d\rho^2 + \rho^2 d\vartheta^2 + \rho^2\sin^2\vartheta d\phi^2\right) + O(\rho^2/R^2)\,, \end{aligned} \tag{2.7}$$

where $C_{\mu\nu}$ are constants that are determined by the specific embedding of the circle in the background. The Riemann tensor of this geometry actually vanishes up to terms $O(R^{-2})$.

Relative to the analysis in [14], we have added new crossed terms $C_{\tau\phi}$ and $C_{z\phi}$ which can appear at the same order. Eq. (2.7) describes the most generic class of backgrounds deformed by S^2-dipole perturbations that preserve the isometries generated by ∂_τ, ∂_z and ∂_ϕ. The value of $C_{\rho\rho}$ can be adjusted at will by an appropriate gauge choice.

In this background, we place a black string of thickness r_0, which will modify the geometry (2.7) at distances $\rho \sim r_0$. Thus the approximations we use will be valid as long as

$$r_0 \ll \min\left(R, \frac{R}{|C_{\mu\nu}|}\right) . \tag{2.8}$$

Eq. (2.1) can be readily evaluated in (2.7), since in this case the extrinsic curvature is simply

$$K_{\mu\nu\sigma} = -\frac{1}{2}\partial_\sigma g_{\mu\nu} \tag{2.9}$$

where μ, ν are parallel to the string (τ and z) and σ is perpendicular. Since, to first order in ρ/R the only coordinate dependence is of dipole type we only have derivatives of the dipolar contribution, $\rho\cos\vartheta$, which evaluated at the location of the string ($\rho = 0, \vartheta = 0$) are purely radial and along the plane in which the ring is curved,

$$\mathcal{F} = \frac{1}{R}\left(C_{\tau\tau}T_{\tau\tau} + 2C_{\tau z}T_{\tau z} - C_{zz}T_{zz}\right) d\rho \, . \tag{2.10}$$

An equilibrium configuration is one for which

$$C_{\tau\tau}T_{\tau\tau} + 2C_{\tau z}T_{\tau z} = C_{zz}T_{zz} \, . \tag{2.11}$$

In the particular case in which the string is boosted to the speed of light, (2.5), this reduces to

$$C_{\tau\tau} + 2C_{\tau z} = C_{zz} \, . \tag{2.12}$$

It is easy to check that this is the same as the equation that determines the null geodesics in (2.7) on the plane $\vartheta = 0$ and at fixed radius $\rho \to 0$. In this case the results are equivalent to more conventional massless probe calculations. But to include non-extremal cases we must resort to the more general approach described above.

2.2 Physical magnitudes

We assume the existence of two commuting Killing vectors that correspond to the canonically-normalized generators of time translation, ζ, and spatial $U(1)$ isometry at infinity, χ. These are related by linear combination to the Killing vectors ∂_τ and ∂_z in the region close to the ring,[5]

$$\begin{aligned} \partial_\tau &= a_0\zeta + b_0\chi \, , \\ \partial_z &= b_1\chi \, . \end{aligned} \tag{2.13}$$

The coefficients a_i, b_i, reflect the possible redshift between the vicinity of the ring and asymptotic infinity, as well as possible rotations and twists between these two regions.

[5]Note that ζ cannot appear in the relation between χ and ∂_z unless we introduce closed timelike curves. Also, in principle other isometries, such as ∂_ϕ, may mix: the discussion below can be easily modified to accommodate this.

The quantities conjugate to these Killing vectors are obtained through integrals of the stress-energy tensor on a spacelike surface Σ with normal n^μ and volume element dV. The mass is conjugate to ζ, so we define

$$M = \int_\Sigma dV T_{\mu\nu} n^\mu \zeta^\nu \,, \tag{2.14}$$

and the momentum conjugate to the $U(1)$ isometry

$$J = \int_\Sigma dV T_{\mu\nu} n^\mu \chi^\nu \tag{2.15}$$

(often this is an angular momentum, but in some instances it is better regarded as linear momentum or electric charge). We can readily obtain the surface gravity and horizon velocity associated to ζ and χ from knowledge of the surface gravity and velocity for ∂_τ and ∂_z. The Killing generator of the horizon is

$$\hat{\xi} = \partial_\tau + \tanh\alpha \partial_z = a_0 \xi \tag{2.16}$$

where

$$\xi = \zeta + \frac{b_0 + b_1 \tanh\alpha}{a_0} \chi \tag{2.17}$$

is the horizon generator in terms of the canonical asymptotic symmetry generators ζ and χ. Thus the horizon velocity relative to infinity is

$$\Omega_H = \frac{b_0 + b_1 \tanh\alpha}{a_0} \,. \tag{2.18}$$

The surface gravity associated to $\hat{\xi}$ is

$$\hat{\kappa} = \frac{1}{2r_0 \cosh\alpha} \tag{2.19}$$

and so the surface gravity measured by asymptotic observers that follow orbits of ξ is

$$\kappa = \frac{1}{2a_0 r_0 \cosh\alpha} \,. \tag{2.20}$$

Finally, the horizon area is computed as the area of the boosted black string,

$$\mathcal{A}_5 = 4\pi r_0^2 \Delta z \cosh\alpha \,. \tag{2.21}$$

In all examples considered so far, these magnitudes are seen to satisfy a first law

$$dM = \frac{\kappa}{8\pi G_5} d\mathcal{A}_5 + \Omega_H dJ \tag{2.22}$$

for variations among stationary solutions when, and only when, the equilibrium conditions are satisfied. We believe this should be generic, and provides a justification for the definitions (2.14) and (2.15).

3 D0-D6 interaction: perturbative methods

We now apply the methods of sec. 2 to study the interaction between D0 and D6 branes in the supergravity approximation. In the absence of other charges, fluxes, or excitations, D0 and D6 branes repel each other. This should be reflected in the non-existence of a supergravity solution that describes them in equilibrium. More precisely, a solution in which the D0 and D6 remain static at a finite distance from each other must contain external forces holding them in place.

However, we expect two ways in which D0 and D6 branes may form bound states at finite separation. The first one has been studied in some detail in the past: a D0 and a D6 brane can form a supersymmetric configuration if an appropriate B-field is turned on in the worldvolume of the D6. When the B-field, which is a modulus, is above a critical value B_c, a bound state between the D0 and D6 appears [24, 26]. A way to understand this effect is by observing that the B-field on the D6 worldvolume induces, through the worldvolume Chern-Simons coupling, Ramond-Ramond fields giving rise to D0, D2, D4 charges. The D2 branes have an attractive effect on the D0, and if the B-field is large enough this attraction may compensate the D6 repulsion. We will be able to study the interaction energy and make explicit how, as the B-field modulus is varied, the potential changes from having no minimum when $B < B_c$, to developing one for $B > B_c$.[6]

A second way in which D0 and D6 branes can be expected to overcome their repulsion is by turning on excitations that increase their gravitational attraction. For simplicity we will only consider thermal excitations of the D0 brane, but in principle it is also possible (and not much more difficult) to excite the D6 brane. Thus we consider a gas of D0 branes with open strings stretching among them in a thermal ensemble. At weak coupling and low energies, this is described by Super-Yang-Mills quantum mechanics at finite temperature, and at higher energies in terms of long excited strings with endpoints on the D0 branes. At strong coupling the description is in terms of a black hole with D0 charge, which in M-theory terms is a black string boosted along the eleventh direction. In the presence of a D6 brane, this becomes a black ring in Taub-NUT.

As in the rest of the paper, we shall take the, more geometrical, M-theory point of view on the system, and thus consider the D0-brane uplifted to M-theory. In a probe approximation, the D0 brane is usually studied as a massless particle moving in a geometry with the structure of a Taub-NUT geometry. However, as discussed at the end of sec.2.1, in order to include also thermal excitations of the D0 brane we must resort to the thin black ring approach developed above.

3.1 Background D6 with B-flux

The background in which we place the D0 brane is that of a D6-brane wrapped on T^6 with a B_{ab} field along its worldvolume directions. For simplicity we shall consider the most symmetric configuration where $B_{12} = B_{34} = B_{56}$, and with a single D6 brane. The construction of the solution, uplifted to M-theory and reduced on T^6 down to five dimensions, is detailed in appendix A. For the purpose of studying the black ring (*i.e.,* the M-uplifted D0 brane) in this background, we shall only need the five-dimensional background metric

$$\begin{aligned} ds^2 &= -Z^{-2}\left[dt + \omega_0 \left(d\psi + (\cos\theta - 1)d\phi\right)\right]^2 \\ &\quad + \frac{Z}{H}\left(d\psi + (\cos\theta - 1)d\phi\right)^2 + ZH(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \end{aligned} \tag{3.1}$$

[6] A threshold case with $B = B_c$ was studied in [30].

with $\psi \sim \psi + 4\pi$, and where

$$H = h + \frac{1}{r}\,, \qquad Z = h_q + \frac{h_p^2}{H}\,, \tag{3.2}$$

and

$$\omega_0 = \frac{3h_p\, h_q}{2H} + \frac{h_p^3}{H^2}\,. \tag{3.3}$$

The moduli at infinity h, h_p, h_q are given by the asymptotic Kaluza-Klein radius L and the B-field, which we express in terms of a dimensionless parameter b as $B = bL/2$

$$h = \frac{2}{L}\frac{1-3b^2}{(1+b^2)^{3/2}}\,, \qquad h_p = \frac{2b}{\sqrt{1+b^2}}\,, \qquad h_q = \frac{L}{2}\sqrt{1+b^2}\,. \tag{3.4}$$

When the B-field vanishes, $b = 0$, we recover the conventional Kaluza-Klein monopole background, with

$$H = \frac{2}{L} + \frac{1}{r}\,, \qquad Z = \frac{L}{2}\,, \qquad \omega_0 = 0 \qquad (b = 0)\,. \tag{3.5}$$

It is well known that this geometry is smooth at the core of the KK monopole ($r = 0$). The same is true for generic values of b. The moduli induce D0-D2-D4 charges, but these do not grow a horizon around the nut (which would require charges not induced by the B-field), nor create a singularity.

We now apply the methods of sec. 2 to this background. First we need the form of the geometry near the location of the ring, which we take to be the circle at

$$r = R\,, \qquad \theta = 0 \tag{3.6}$$

extended along ψ. Let us denote

$$H_R \equiv H(r = R)\,, \qquad Z_R \equiv Z(r = R)\,, \qquad \omega_R \equiv \omega_0(r = R)\,. \tag{3.7}$$

The proper circumferential length of the circle is

$$\Delta z = \int d\psi \sqrt{g_{\psi\psi}}|_{r=R,\theta=0} = 2\pi L \frac{\sqrt{2R\left((1+b^2)^{3/2}L + 2R\right)}}{\sqrt{1+b^2}L + 2R}\,. \tag{3.8}$$

As the distance between the ring and the nut grows, $R \to \infty$, this becomes equal to the asymptotic KK circle length

$$\Delta z \to 2\pi L\,. \tag{3.9}$$

Observe that R is not the proper radial distance between the ring and the nut but only a coordinate distance associated to the conventional cooodinate r in (3.1). However, we will continue to use it as a simple and convenient measure of the separation between the D0 and D6 brane.

In order to focus on the region around the circle (3.6), we change to adapted coordinates $(r, \theta) \to (\rho, \vartheta)$

$$r \sin\theta = \frac{\rho}{\sqrt{H_R Z_R}} \sin\vartheta, \qquad r\cos\theta = R + \frac{\rho}{\sqrt{H_R Z_R}}\cos\vartheta \tag{3.10}$$

such that the ring circle (3.6) now lies at $\rho = 0$, and then expand the metric in powers of ρ/R. To bring the metric into the form (2.3) to zero-th order in ρ/R, we have to perform two further

coordinates changes: first, change to corotating coordinates, and then rescale time to canonical normalization and ψ to proper length along the string direction,

$$t=\frac{\Delta z}{4\pi}\sqrt{Z_R H_R}\,\tau, \qquad \psi=\frac{4\pi}{\Delta z}\left(z+\sqrt{\frac{H_R}{Z_R^3}}\omega_R\,\tau\right). \tag{3.11}$$

Now the metric to first order in ρ/R takes the form (2.7), with

$$\begin{aligned}
C_{\tau\tau}=&\frac{Lb^2}{(1+b^2)^{3/2}L+2R}\sqrt{\frac{2(1+b^2)R^3}{\left(\sqrt{1+b^2}L+2R\right)^3}}\\
C_{\tau z}=&\frac{2b^3LR^2}{\left(\sqrt{1+b^2}L+2R\right)^2((1+b^2)^{3/2}L+2R)}\\
C_{zz}=&\frac{L\left((1+b^2)^{3/2}L+2(1-b^2)R\right)}{(1+b^2)^{3/2}L+2R}\sqrt{\frac{(1+b^2)R}{2\left(\sqrt{1+b^2}L+2R\right)^3}}\\
C_{\rho\rho}=&-L\sqrt{\frac{(1+b^2)R}{2\left(\sqrt{1+b^2}L+2R\right)^3}}\\
C_{z\phi}=&-\frac{L\sqrt{2R\left((1+b^2)^{3/2}L+2R\right)}}{4\left(\sqrt{1+b^2}L+2R\right)^2},
\end{aligned} \tag{3.12}$$

and $C_{\tau\phi}=0$.

The thin ring approximation is valid when (2.8) holds. When $b=0$ this condition is always parametrically equivalent to simply

$$r_0 \ll R\,. \tag{3.13}$$

Observe that when the ring is far from the nut, $R \gg L$, the approximation is also valid for $r_0 > L$: in this regime, in which the black ring is very well approximated by a wrapped black string, the thickness r_0 is only limited by the requirement that the ring remains away from the nut. The ring thickness itself can be much larger than the KK radius. This also remains valid with non-zero b, since a large B-field $b \gg 1$ tends to make the coefficients $C_{\mu\nu}$ smaller.

3.2 Physical parameters

We need to know how the parameters r_0 and α relate to the number, mass and entropy of D0 branes. To find this we need the relation between the Killing generators ∂_τ, ∂_z in the region near the ring, and the canonical generators χ of the asymptotic compact circles with period $2\pi L$, and ζ of asymptotic time translations.

To this effect (see [16]), we first note that the metric at asymptotic infinity becomes

$$ds^2 \to \frac{L^2}{4}\left(d\psi+(\cos\theta-1)d\phi-\varpi d\bar{t}\right)^2-d\bar{t}^2+dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{3.14}$$

where $\bar{t}=2t/L$ is the canonically normalized time and

$$\varpi=\frac{2}{L}\frac{b(3-b^2)}{(1+b^2)^{3/2}} \tag{3.15}$$

is the velocity $d\psi/d\bar{t}$ of the asymptotic frames. We now change

$$\psi = \frac{2}{L}y + \varpi\bar{t} + \bar{\phi}\,, \qquad \phi = \bar{\phi} \tag{3.16}$$

in order to go to the canonical asymptotic form for the KK vacuum in its rest frame,

$$ds^2 \to \left(dy + \frac{L}{2}\cos\theta\, d\bar{\phi}\right)^2 - d\bar{t}^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\bar{\phi}^2 \tag{3.17}$$

with $y \sim y + 2\pi L$.

The timelike Killing generator, ζ, of the orbits of static asymptotic observers is[7]

$$\zeta = \frac{\partial}{\partial\bar{t}} = \frac{L}{2}\frac{\partial}{\partial t} + \varpi\frac{\partial}{\partial\psi}\,, \tag{3.18}$$

the generator χ of the Kaluza-Klein circle is

$$\chi = \frac{\partial}{\partial y} = \frac{2}{L}\frac{\partial}{\partial\psi}\,, \tag{3.19}$$

and the angular rotations along $\bar{\phi}$ are generated by

$$\frac{\partial}{\partial\bar{\phi}} = \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}\,. \tag{3.20}$$

Now, since eq. (3.11) gives

$$\frac{\partial}{\partial\psi} = \frac{\Delta z}{4\pi}\frac{\partial}{\partial z}\,, \qquad \frac{\partial}{\partial t} = \frac{4\pi}{\Delta z}\frac{1}{\sqrt{H_R Z_R}}\left(\frac{\partial}{\partial\tau} - \sqrt{\frac{H_R}{Z_R^3}}\omega_R\frac{\partial}{\partial z}\right)\,, \tag{3.21}$$

then together with (3.18) and (3.19) we obtain the relations we sought.

We can now compute the physical magnitudes following the analysis in sec. 2.2. We shall express them as four-dimensional quantities, taking into account that

$$G_4 = \frac{G_5}{2\pi L}\,. \tag{3.22}$$

The magnetic charge comes entirely from the background D6 and is

$$P = \frac{L}{4}\,, \tag{3.23}$$

and the 4D electric charge is proportional to the momentum along the compact direction,

$$Q = 2G_4\int dz d^3x T_{\tau\mu}\chi^\mu = 2G_4\frac{N_0}{L} \tag{3.24}$$

where N_0 is given in (2.6).

[7]Note that ζ differs from the timelike Killing vector ∂_t that is parallel to the supersymmetry generators (*i.e.*, constructed as a bilinear of Killing spinors).

Since the black string has no angular momentum along ϕ, it follows from (3.20) that the four-dimensional angular momentum associated to $\partial_{\bar{\phi}}$ is given by the Dirac value

$$J = \frac{QP}{G_4}\,. \tag{3.25}$$

If we had we taken the black string to be a Kerr black string then $J \neq \frac{QP}{G_4}$ — this is illustrated in appendix E. This is only possible when the D0 is excited, since in the limit $\alpha \to \infty$ the S^2 rotation must vanish to avoid pathologies, and so the Dirac relation must be satisfied.

The energy conjugate to time translations generated by ζ is

$$M = \int dz d^3x T_{\tau\mu}\zeta^\mu = \frac{Q}{2G_4}\left[\left(\frac{2\pi L}{\Delta z}\right)^2\left(\frac{1}{\sqrt{H_R Z_R}}\frac{\cosh^2\alpha+1}{\sinh\alpha\cosh\alpha} - \frac{\omega_R}{Z_R^2}\right) + \frac{L\varpi}{2}\right]. \tag{3.26}$$

The four-dimensional Einstein-frame area is

$$\mathcal{A}_4 = \frac{\Delta z}{2\pi L}4\pi r_0^2\cosh\alpha\,, \tag{3.27}$$

consistently with the invariance of the entropy under dimensional reduction. When the ring is non-extremal and α is finite, we can express the area for a given electric charge (*i.e.*, the entropy for a given net number of D0s) as

$$\frac{\mathcal{A}_4}{16\pi Q^2} = \frac{1}{\sinh^2\alpha\cosh\alpha}\left(\frac{2\pi L}{\Delta z}\right)^3, \tag{3.28}$$

where Δz is given in (3.8).

3.3 Equilibrium configurations

Equilibrium configurations correspond to solutions of (2.11). For a black ring (2.4) this requires

$$(\cosh^2\alpha+1)C_{\tau\tau} + 2C_{\tau z}\cosh\alpha\sinh\alpha = C_{zz}(\sinh^2\alpha - 1) \tag{3.29}$$

with $C_{\mu\nu}$ given by (3.12). There are two simple situations of particular interest:

Extremal ring with $B \neq 0$

When the ring is boosted to the speed of light, $\alpha \to \infty$ —so it is extremal and chiral, corresponding to a D0 brane in its ground state— the equilibrium equation (2.12) is solved for

$$R = \frac{L}{2}\frac{(1+b^2)^{3/2}}{3b^2-1}. \tag{3.30}$$

Thus, equilibria between D0 and D6 branes are possible for

$$b > b_c = \frac{1}{\sqrt{3}}\,. \tag{3.31}$$

In appendix B we compare the result (3.30) with the one obtained from the exact supergravity solution, and show perfect agreement when the D0 branes are treated perturbatively. The critical value of the field (3.31) also agrees with the value computed in perturbative string theory [26, 24].

Non-extremal ring with $B = 0$

When the D0 branes are thermally excited the boost α is finite. In this case we can look for equilibrium configurations even when no B flux is present, $b = 0$. This simplifies greatly the background, since $C_{\tau\tau} = C_{\tau z} = 0$ and the equilibrium condition fixes the boost value to

$$\sinh^2 \alpha = 1\,. \tag{3.32}$$

In this case, we can write (3.28) (with $b = 0$ in (3.8)) as

$$R = \frac{L}{2}\left[\left(\frac{\mathcal{A}_4}{8\sqrt{2}\pi Q^2}\right)^{2/3} - 1\right]^{-1} \tag{3.33}$$

which implies that, for a fixed net number of D0 branes (fixed Q), a bound state can exist if the thermal excitation is large enough to create a horizon of area

$$\mathcal{A}_4 > \mathcal{A}_{4,c} = 8\sqrt{2}\pi Q^2\,, \tag{3.34}$$

or in terms of entropy and D0 number,

$$S > S_c = 8\sqrt{2}\pi G_4 \frac{N_0^2}{L^2}\,. \tag{3.35}$$

It is important to observe that, even if we have derived this result using the perturbative method for thin rings, the result (3.35) is actually *exactly* valid, since it is the value of the entropy for a black ring bound at an infinite distance from the nut, in which case (3.13) does not impose any constraint on the ring thickness.

For $b = 0$ and equilibrium boost (3.32), the expressions for the physical parameters take simple forms,

$$\frac{G_4 M}{Q} = \frac{3}{2\sqrt{2}}\left(\frac{2\pi L}{\Delta z}\right), \qquad \frac{\mathcal{A}_4}{(G_4 M)^2} = \frac{64\sqrt{2}\pi}{9}\left(\frac{2\pi L}{\Delta z}\right). \tag{3.36}$$

Since $\Delta z \leq 2\pi L$, with saturation when the separation goes to infinity, we see that not only the area, but also the mass has a lower limit

$$M > M_c = \frac{3}{2\sqrt{2}}\frac{Q}{G_4} = \frac{3}{\sqrt{2}}\frac{N_0}{L} \tag{3.37}$$

for the bound state to exist.

Thus we have demonstrated the two main mechanisms that permit the formation of bound states of D0 and D6 branes. The general case in which the two are simultaneously at work, *i.e.,* when both b and r_0 are finite, is only technically more difficult and not more illuminating, so we will not dwell on it.

3.4 Forces, interaction energy, and stability

In a general configuration away from equilibrium, an external force $\mathcal{F}$ is needed in order to keep the D0 branes in place, which acts at each point along the ring. Thus the ring exerts a force $-\mathcal{F}$, and we can assign to it a potential energy $dV_{\mathcal{F}} = \sqrt{-g_{tt}}\,\Delta z\,\mathcal{F}$, redshifted from the location of the ring to asymptotic infinity. Thus we introduce

$$V_{\mathcal{F}}(R) = -\int_R^\infty \sqrt{-g_{tt}}\,\Delta z\,\mathcal{F}\,. \tag{3.38}$$

Clearly, equilibrium corresponds to $V'_{\mathcal{F}} = 0$.

There is another measure of the energy of the interaction, given by shift in the *internal* energy of the D0-branes when placed in the field created by the D6 brane—the D6 is not affected since we are regarding it as a background. This shift is the difference between the measured mass M of the D0s in the presence of the D6 and their mass in isolation,

$$\Delta M(R) = M(R) - M_{\mathrm{D0}}\,. \tag{3.39}$$

By the mass of isolated D0s we mean the mass of a D0-charged black hole of given charge and area. Then ΔM measures the change in its internal energy as it is moved adiabatically from infinity to R. When the D0 is extremal, its mass in isolation is simply $Q/(2G_4)$. We mentioned above that the mass is extremized for configurations that satisfy the no-force condition. Thus, the extrema of ΔM coincide with the extrema of $V_{\mathcal{F}}$.

In principle ΔM and $V_{\mathcal{F}}$ seem to be different contributions to the total interaction energy, and we might expect that the quantity that corresponds to E_{int} in (1.2), which we use for exact solutions, is the sum of both. In fact we have checked explicitly that, at large distances, the value of E_{int} for the exact solutions coincides with that of $\Delta M + V_{\mathcal{F}}$.

On the other hand, not only do the minima of ΔM and $V_{\mathcal{F}}$ coincide, but the two functions also resemble each other closely for generic b and r_0, and in fact agree exactly for $b = 0$. Whether adding up ΔM and $V_{\mathcal{F}}$ is the correct procedure, or instead is double-counting the interaction energy, is not completely clear to us,[8] but fortunately none of our conclusions depends on this, since the properties of the interaction energy remain the same (up to possibly a factor close to 2) with either definition. The interaction energy $E_{\mathrm{int}} = \Delta M + V_{\mathcal{F}}$ for the two particular cases of interest discussed above has been presented in figs. 2 and 3. Here we discuss some of their properties at short and long distances.

If we consider first $R \to 0$ it is easy to see that in the extremal case at fixed charge, the mass M in (3.26) diverges as $1/\sqrt{R}$, and so does then, too, ΔM. Thus at short distances we cannot expect our perturbative approximation to remain valid and we must resort to solutions that account for backreaction of the D0. This is addressed in sec. 5.

At large distances the values of ΔM and $V_{\mathcal{F}}$ are equal to leading order. Adding them together to obtain the total interaction energy we find

$$G_4 E_{\mathrm{int}}(R) = -\left(2b^3 + \sqrt{1+b^2}\,\frac{(2b^2-1)\sinh^2\alpha_\infty + b^2 + 1}{\cosh\alpha_\infty \sinh\alpha_\infty}\right)\frac{QP}{R} + O(R^{-2})\,, \tag{3.40}$$

where α_∞ is a function of $Q^2/\mathcal{A}_4$, independent of b, determined as the solution of (3.28) at $R \to \infty$, *i.e.*, $\Delta z \to 2\pi L$,

$$\sinh^2\alpha_\infty \cosh\alpha_\infty = \frac{16\pi Q^2}{\mathcal{A}_4}\,. \tag{3.41}$$

Depending on the sign of the coefficient of QP/R the interaction energy will be repulsive or attractive. This is in fact in line with the force analysis, since this coefficient is proportional to the force $\mathcal{F}$ at $R \to \infty$, and when it vanishes a bound state at infinity appears. We can see this more explicitly in the two particular cases of interest. For the extremal case, $\alpha_\infty \to \infty$ ($\mathcal{A}_4 \to 0$) the above expression simplifies to

$$G_4 E_{\mathrm{int}}(r) = \frac{1-3b^2}{2b^3 - \sqrt{1+b^2}(2b^2-1)}\frac{QP}{R} + O(R^{-2})\,. \tag{3.42}$$

[8] The fact that this prescription would give a larger total mass to the system at equilibrium than $M_{\mathrm{D6}} + M$ points in this direction.

This interaction energy changes from positive, hence repulsive, for $b < 1/\sqrt{3}$ to negative, hence attractive, for $b > 1/\sqrt{3}$. The critical value $b_c = 1/\sqrt{3}$ agrees with the expectation that the long-distance perturbative string interaction changes sign at $b = b_c$. At $b = 0$ it becomes particularly simple,

$$G_4 E_{\rm int}(R) = \frac{QP}{R} + O(R^{-2})\,. \tag{3.43}$$

For the nonextremal ring with $b = 0$ we have

$$G_4 E_{\rm int}(r) = \frac{\sinh^2\alpha_\infty - 1}{\cosh\alpha_\infty \sinh\alpha_\infty}\frac{QP}{R} + O(R^{-2})\,. \tag{3.44}$$

We see that bound a state at infinity appears for $\sinh^2\alpha_\infty = 1$ as expected, and that the interaction becomes attractive when the ring at infinity is rotating more slowly, $\sinh^2\alpha_\infty < 1$: the centrifugal force at infinity is weaker so the ring will tend to shrink and move towards the nut.

4 Exact solution-generating technique

In the case of five-dimensional stationary vacuum solutions with two $U(1)$ isometries, one can actually go beyond the perturbative approximation and construct exact geometries by the application of a combination of solution-generating techniques. In particular, we will use the method of [10] and [12] to generate an asymptotically Taub-NUT solution starting from an asymptotically flat solution. This method is based on the application of an appropriate element of the $SL(3,\mathbb{R})$ group of symmetries, discovered in [33], that the solutions to the Einstein vacuum equations with these isometries have. In this section, after briefly recalling the solution generating technique that we employ, we will show how to construct an exact solution corresponding to a black ring in Taub-NUT.

4.1 Review of the solution-generating method

The starting solution, that will be referred to as the "seed", is a stationary axisymmetric solution ([34]) in five dimensions:

$$ds^2 = G_{IJ}dy^I dy^J + e^{2\nu}(d\rho^2 + dz^2)\,, \tag{4.1}$$

where y^I, $I = 0, 1, 2$, are coordinates corresponding to Killing directions of the solution; in our case $y^I = \{t, \hat\phi, \hat\psi\}$, with $\hat\phi$ and $\hat\psi$ the Cartan angles of $\mathbb{R}^4$. Here the metric coefficients G_{IJ} and $e^{2\nu}$ only depend on the Weyl coordinates ρ and z. The $SL(3,\mathbb{R})$ transformation that relates asymptotically flat and Taub-NUT solutions acts naturally on the Euler angles $\phi_\pm$:

$$\hat\psi = \frac{1}{2}(\phi_+ + \phi_-)\,, \qquad \hat\phi = \frac{1}{2}(\phi_+ - \phi_-)\,. \tag{4.2}$$

We will identify the fiber of Taub-NUT space with the direction ϕ_+. Introducing coordinates $\xi^0 \equiv t$ and $\xi^1 = \ell\phi_+$, where ℓ is an arbitrary length scale, it is useful to rewrite the metric (4.1) in the form

$$ds^2 = \lambda_{ab}(d\xi^a + \omega^a)(d\xi^b + \omega^b) + \frac{1}{\tau}\,ds_3^2\,, \tag{4.3}$$

where $a,b=0,1$, $\tau=-\det\lambda_{ab}$ and $\omega^a=\omega^a_- d\phi_-$ are one-forms on the base space ds_3^2. The three-dimensional metric ds_3^2 on the base space is then given by

$$ds_3^2=\tau\, e^{2\nu}(d\rho^2+dz^2)+\frac{\rho^2}{4\,\tau}\, d\phi_-^2\,. \tag{4.4}$$

Using the fact that in three dimensions a one-form is dual to a scalar, we can introduce the potentials V_a:

$$dV_a=-\tau\lambda_{ab}\ast_3 d\omega^b\,, \tag{4.5}$$

where the Hodge operation $\ast_3$ is performed with the metric ds_3^2. It can be shown that the integrability condition of this equation is satisfied thanks to the Einstein equations. Then, the data contained in the metric (4.3) can be re-packadged into the symmetric unimodular matrix of scalars χ:

$$\chi=\begin{pmatrix}\lambda_{ab}-\frac{V_aV_b}{\tau} & \frac{V_a}{\tau}\\ \frac{V_b}{\tau} & -\frac{1}{\tau}\end{pmatrix}. \tag{4.6}$$

The usefulness of this formalism relies on the fact that the equations of motion are left invariant by the action of an $SL(3,\mathbb{R})$ group of transformations that act linearly on χ:

$$\chi\to\chi'=N\,\chi\,N^T\,,\qquad ds_3^2\to ds_3^2\,,\qquad N\in SL(3,\mathbb{R})\,. \tag{4.7}$$

This provides a solution generating method: starting from a solution (χ, ds_3^2), one can construct a new solution (χ', ds_3^2) by acting on the former with suitable elements of $SL(3,\mathbb{R})$. Reconstruting the final metric from the rotated matrix χ' requires inverting the duality relations (4.5) in order to compute the transformed one-forms ω'^a. As a computational trick to facilitate this procedure, it is useful to introduce a matrix of one-forms κ, defined as

$$\ast_3\, d\kappa=\chi^{-1}d\chi\,. \tag{4.8}$$

One can show that κ encodes the information about the one-forms ω^a since

$$\omega^a=-\kappa^a{}_2\quad (a=0,1)\,. \tag{4.9}$$

Moreover, the definition of κ implies that it also transforms linearly under $SL(3,\mathbb{R})$:

$$\kappa\to\kappa'=(N^{-1})^T\,\kappa\,N^T\,,\qquad N\in SL(3,\mathbb{R})\,. \tag{4.10}$$

Therefore, by following the transformations of both χ and κ under $SL(3,\mathbb{R})$, one can easily reconstruct the transformed metric by purely algebraic manipulations. We will refer to the set of data (χ,κ) as the Maison data.

In [10], [12] the element of $SL(3,\mathbb{R})$ that maps five-dimensional asymptotically flat solutions into asymptotically Taub-NUT solutions was identified. To construct a black ring in Taub-NUT, one should, naively, apply this $SL(3,\mathbb{R})$ transformation to the black ring of [5]. However, as explained in [12], this does not quite work: the $SL(3,\mathbb{R})$ transformations change the relative orientation of the rods which can spoil the regularity of the solution. The application of an $SL(3,\mathbb{R})$ transformation to a solution with a regular horizon of topology $S^2\times S^1$ produces, in general, a solution with a singular horizon. To counterbalance this effect, one should start from a singular seed solution that generalizes the flat space black ring and contains an extra parameter encoding the relative orientation of the space-like rods on either side of the horizon.

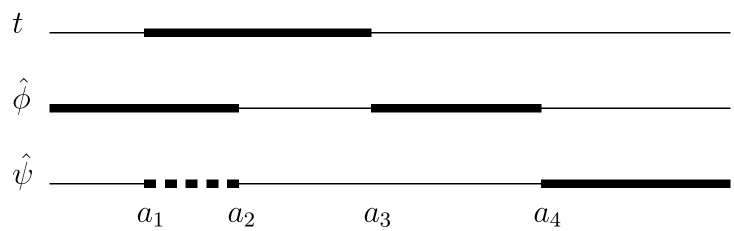

Figure 4: Rod structure of the initial solution (4.11). This is the same rod structure of the seed solution of the standard S^1–spinning ring.

This extra parameter is then fixed in such a way that the $SL(3,\mathbb{R})$-transformed solution has a regular ring-like horizon.

The easiest way to produce the needed seed solution is to use the BZ [35] (see [36] for a detailed review) technique to construct a one-parameter family of solutions that generalizes the black ring of [5]. Given a solution generated by the BZ method, a series of technical results derived in [12] allow to compute the corresponding χ and κ matrices. In section 4.2 we will construct the appropriate seed and compute the associated Maison data. The $SL(3,\mathbb{R})$ transformation will be carried out in section 4.3, where we will also perform a singularity analysis and determine the values of parameters for which the solution is regular.

4.2 Seed solution

In this section we construct the appropriate seed solution to which we will apply a suitable $SL(3,\mathbb{R})$ transformation that will eventually yield a black ring in Taub-NUT. The data encoding the seed solution are the metric factors $\{G_{IJ},\, e^{2\nu}\}$ defined in (4.1).

The metric

Our starting point is a Weyl solution given by

$$G_0 = \mathrm{diag}\left\{-\frac{\mu_1}{\mu_3}, \frac{\rho^2\mu_3}{\mu_2\mu_4}, \frac{\mu_2\mu_4}{\mu_1}\right\}\,. \tag{4.11}$$

The first term in G_0 corresponds to the tt-component, the second to the $\hat\phi\hat\phi$-component and the third to the $\hat\psi\hat\psi$-component. In additon, we have

$$e^{2\nu_0} = k^2\, \frac{\mu_2\mu_4(\rho^2+\mu_1\mu_2)(\rho^2+\mu_1\mu_3)(\rho^2+\mu_1\mu_4)(\rho^2+\mu_2\mu_3)(\rho^2+\mu_3\mu_4)}{\mu_1(\rho^2+\mu_2\mu_4)^2\prod_{i=1}^4(\rho^2+\mu_i^2)}\,, \tag{4.12}$$

with $k>0$ without loss of generality. We use the standard notation

$$\mu_i = \sqrt{\rho^2+(z-a_i)^2}-(z-a_i)\,, \qquad \bar\mu_i = -\rho^2/\mu_i\,, \qquad i=1\ldots 4 \tag{4.13}$$

and we assume the ordering

$$a_1 \le a_2 \le a_3 < a_4\,, \tag{4.14}$$

of the rod endpoints. The initial solution (4.11)-(4.12) is singular by itself, but as explained in [8], the singularity can be canceled after the soliton transformations by fixing the BZ parameters conveniently.

We generate the wanted seed solution by means of a two-soliton transformation on (4.11). We briefly summarize the steps of this construction:

1. Remove an anti-soliton at $z = a_1$ and a soliton at $z = a_4$ form $(G_0)_{tt}$ and $(G_0)_{\hat{\phi}\hat{\phi}}$ respectively. The resulting metric is:

$$G_1 = \text{diag}\left\{ \frac{\rho^2}{\mu_1\mu_3}, -\frac{\mu_3\mu_4}{\mu_2}, \frac{\mu_2\mu_4}{\mu_1} \right\} . \tag{4.15}$$

2. Rescale the metric by a factor of $1/\mu_4$ to find

$$\tilde{G}_0 = \frac{1}{\mu_4} G_1 = \text{diag}\left\{ -\frac{\bar{\mu}_3}{\mu_1\mu_4}, -\frac{\mu_3}{\mu_2}, \frac{\mu_2}{\mu_1} \right\} . \tag{4.16}$$

This is the new seed solution to which we apply the BZ transformations. The corresponding generating matrix is then given by

$$\Psi_0 = \text{diag}\left\{ -\frac{(\bar{\mu}_3 - \lambda)}{(\mu_1 - \lambda)(\mu_4 - \lambda)}, -\frac{(\mu_3 - \lambda)}{(\mu_2 - \lambda))}, \frac{(\mu_2 - \lambda)}{(\mu_1 - \lambda)} \right\} . \tag{4.17}$$

3. Perform now a two-soliton transformation with $\tilde{G}_0$ as seed, re-adding the anti-soliton at $z = a_1$ and the soliton at $z = a_4$ with BZ vectors $m_0^{(1)} = (1, 0, b_1)$, $m_0^{(4)} = m(0, 1, b_4)$ respectively. Denote the resulting solution by $\tilde{G}$.

4. Rescale $\tilde{G}$ to find the final metric:

$$G = \mu_4 \tilde{G} . \tag{4.18}$$

Note that by construction G satisfies $\det G = -\rho^2$.

The solution obtained at this stage has a naked singularity at $z = a_1$, the position of the negative density rod of the starting metric G_0. This singularity can be removed by fixing the BZ parameter b_1 to be

$$|b_1| = \sqrt{\frac{2(a_2 - a_1)(a_4 - a_1)}{(a_3 - a_1)}} . \tag{4.19}$$

From now on we will only consider the solution with b_1 fixed as above. The sign of b_1 is arbitrary and it can always be changed by changing $\hat{\psi} \to -\hat{\psi}$, which corresponds to inverting the sense of rotation of the five-dimensional asymptotically flat solution. Without loss of generality, we choose $b_1 > 0$.

Parametrization

Having fixed b_1, the resulting solution can be conveniently parametrized so that its relation with the five-dimensional asymptotically flat black ring is manifest. Following [8], we choose

$$a_1 = -R^2 \frac{2\lambda - \nu(1 + \lambda)}{2(1 - \lambda)}, \qquad a_2 = -\frac{R^2}{2}\nu \qquad a_3 = \frac{R^2}{2}\nu , \qquad a_4 = \frac{R^2}{2} . \tag{4.20}$$

Notice that in this paramtrization b_1 is given by

$$b_1 = R\sqrt{\frac{(1 + \lambda)(\lambda - \nu)}{\lambda(1 - \lambda)}} . \tag{4.21}$$

We find it convenient to change to the C-metric type of coordinates of [37], for which

$$\rho = \frac{R^2}{(x-y)^2}\sqrt{-G(x)G(y)}\,, \qquad z = \frac{R^2(1-xy)\big[1+\nu(x+y)\big]}{(x-y)^2}\,, \tag{4.22}$$

where $G(\xi)$ is defined below. Notice that, defining $R_i = \sqrt{\rho^2 + a_i^2}$, $i = 2, 3, 4$, we can invert the relations above and write (x, y) in terms of (ρ, z):

$$x = \frac{(1-\nu)R_2 - (1+\nu)R_3 - 2R_4 + R^2(1-\nu^2)}{(1-\nu)R_2 + (1+\nu)R_3 + 2\nu R_4}\,, \tag{4.23a}$$

$$y = \frac{(1-\nu)R_2 - (1+\nu)R_3 - 2R_4 - R^2(1-\nu^2)}{(1-\nu)R_2 + (1+\nu)R_3 + 2\nu R_4}\,. \tag{4.23b}$$

Finally, rescaling b_4 as $b_4 = \bar{b}_4(1+\nu)^2\sqrt{\frac{1-\lambda}{1+\lambda}}$, we can write the metric (4.18) in a simple looking form,[9]

$$\begin{aligned} ds^2 = &-\frac{H(y,x)}{H(x,y)}\Big[dt+\Omega\Big]^2 \\ &+\frac{R^2}{(x-y)^2}\left[-\frac{F(y,x)}{H(y,x)}d\hat{\psi}^2 + k^2\,H(x,y)\left(-\frac{dy^2}{G(y)}+\frac{dx^2}{G(x)}\right)+\frac{F(x,y)}{H(y,x)}d\hat{\phi}^2+\frac{2J(x,y)}{H(y,x)}d\hat{\phi}d\hat{\psi}\right], \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} G(x) &= (1-x^2)(1+\nu x)\,, && (4.25)\\ H(x,y) &= 1+\lambda x - \bar{b}_4^2(1+\nu x)^2(1+\nu y)\big[1-\lambda\nu-(\lambda-\nu)y\big]\,, && (4.26)\\ F(x,y) &= G(x)(1+\lambda y) - \bar{b}_4^2 G(y)(1+\nu x)^3\big[1-\lambda\nu-(\lambda-\nu)x\big]\,, && (4.27)\\ J(x,y) &= -(1+\lambda)\bar{b}_4\,C_2\,(x-y)(1+\nu x)(1+\nu y)\big[x+y+\nu(1+xy)\big]\,, && (4.28) \end{aligned}$$

and the rotation one-form Ω is given by

$$\Omega = \frac{R\,C_1}{H(y,x)}\big[\omega_{\hat{\psi}}(x,y)d\hat{\psi} + \bar{b}_4\,C_2\,\omega_{\hat{\phi}}(x,y)d\hat{\phi}\big]\,, \tag{4.29}$$

with

$$\begin{aligned} \omega_{\hat{\psi}}(x,y) &= 1+y-\bar{b}_4^2(1-\nu)(1-x)(1+\nu x)(1+\nu y)^2\,, && (4.30)\\ \omega_{\hat{\phi}}(x,y) &= (1+\nu x)\big[x+y+\nu(1+xy)\big]\,, && (4.31) \end{aligned}$$

and

$$C_1 = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,, \qquad C_2 = \sqrt{\frac{1-\lambda}{1+\lambda}}\,. \tag{4.32}$$

Note that we have left the constant k in front of the conformal factor unspecified. We will fix it later on when we consider the asymptotics of the final metric. Finally, the dimensionless parameters λ and ν must lie in the range

$$0 < \nu \le \lambda < 1\,. \tag{4.33}$$

[9]This metric is a particular case of the solution constructed in [38]. However at this stage we have not imposed any condition on the parameters that determine the directions of the rods.

The metric (4.24) is not written in a manifestly asymptotically flat form. Though one could perform a change of coordinates to bring the metric in an asymptotically flat frame, this is not needed for our construction. Furthermore, the solution (4.24) has closed timelike curves due to the fact that the direction of the finite spacelike rod $z \in [a_3, a_4]$, has a component along t, which implies that t has to be globally identified with a certain period. In fact, this corresponds to the presence of a Dirac-Misner string. As explained above, this problem will be cured after the action of $SL(3, \mathbb{R})$. Finally, we notice that setting $b_4 = 0$, or equivalently $\bar{b}_4 = 0$, the metric (4.24) reduces to the S^1–spinning ring of [37].

To apply the $SL(3, \mathbb{R})$ transformation that will generate the metric for a black ring in Taub-NUT, one needs to compute the Maison data (χ, κ) for the metric (4.24). This computation is rather involved and the interested reader can find it in appendix C.

4.3 Constructing the ring

To proceed with the construction of the solution, we examine first the rod structure [34] of the seed solution (4.24). There are the following four rods: rod 1, at $z \in (-\infty, -\frac{R^2}{2}\nu]$ (or $x = -1$); rod 2, at $z \in [-\frac{R^2}{2}\nu, \frac{R^2}{2}\nu]$ (or $y = -\frac{1}{\nu}$); rod 3, at $z \in [\frac{R^2}{2}\nu, \frac{R^2}{2}]$ (or $x = 1$); rod 4, at $z \in [\frac{R^2}{2}, \infty)$ (or $y = -1$). As shown in [10], the eigenvectors, v_i ($i = 1, \ldots, 4$), associated to each rod can be easily derived from the matrix κ as

$$v_i = \lim_{\rho \to 0} (\kappa_{02}, \kappa_{12}, 1)\Big|_{z \in I_i}, \tag{4.34}$$

where I_i is the interval corresponding to the i-th rod, and we are writing the vectors in the basis $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \xi^1}, \frac{\partial}{\partial \phi_-}\}$. This result holds whenever $\tau \neq 0$, which is satisfied on every rod for our seed solution.

The rod 2 is timelike and therefore corresponds to the horizon of the solution. In order for the topology of this horizon be $S^2 \times S^1$, it is necessary that its neighboring rods, namely rods 1 and 3, have the same direction, i.e. $v_1 = v_3$. One can check, using (4.34), that our seed solution does not satisfy this requirement and hence it is not a black ring.[10] As explained above, however, this feature of the seed solution is exactly what is needed to produce a regular black ring in Taub-NUT: the application of a suitable $SL(3, \mathbb{R})$ transformation will add KK-monopole charge and, at the same time, modify the relative orientation of rods 1 and 3 in such a way that the final solution will satisfy $v_1 = v_3$. Let us see how this works in some detail.

To add KK-monopole charge to an asymptotically flat solution, one should apply to the seed solution the transformation D, where D is a particular element of $SL(3, \mathbb{R})$ (see below) [10]. However, the action of D generates unwanted Dirac-Misner strings. This pathology can be canceled by further acting on the solution with an element in the $SO(2, 1)$ subgroup of $SL(3, \mathbb{R})$ that preserves the asymptotic boundary conditions. There are three such transformations, denoted as N_α, N_β and N_γ in [20]. N_α is equivalent to a "boost" in the ξ^1 direction. Such a transformation does not change the relative orientation of the rods, nor the periodicity of the angular coordinates and, for this reason, does not affect the topology of the horizon. It has been shown in [10] that the action N_β is equivalent to a redefinition of the scale ℓ and hence, if one keeps ℓ as an arbitrary parameter, the action of N_β is superfluous. On the other hand, N_γ changes the regularity properties of the geometry, and we will need to include its action to obtain the desired solution.

[10] Instead, this solution can be interpreted as black lens [38].

In a first step we obtain an asymptotically Taub-NUT solution by acting on the seed (4.24) with D and N_γ. In terms of the Maison data (χ', κ'), the new solution is given by

$$\chi' = N_\gamma D\, \chi\, D^T N_\gamma^T\,, \quad \kappa' = N_\gamma D\, \kappa\, D^T N_\gamma^T\,, \tag{4.35}$$

where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\,, \quad N_\gamma = \begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix}\,. \tag{4.36}$$

The resulting metric has rods at the same positions as the seed metric but with different orientations, which are determined by the parameters of the $SL(3,\mathbb{R})$ transformation. The eigenvectors corresponding to the rods 1 and 3 of the solution in (4.35) have the form

$$\begin{aligned} v_1' &= \Big(\ell(c_1^0 \cos 2\gamma + s_1^0 \sin 2\gamma), \ell(c_1^1 \cos\gamma + s_1^1 \sin\gamma), 1\Big)\,, \\ v_3' &= \Big(\ell(c_3^0 \cos 2\gamma + s_3^0 \sin 2\gamma + z_3^0), \ell(c_3^1 \cos\gamma + s_3^1 \sin\gamma), 1\Big)\,, \end{aligned} \tag{4.37}$$

where c_j^i, s_j^i and z_3^0 are some functions of b_4, λ, ν and the dimensionless ratio

$$\hat{R} \equiv \frac{R}{\ell}\,. \tag{4.38}$$

We should require that $v_1' = v_3'$ in order for the new solution be a black ring. The ξ^1-component of this equation fixes the angle γ in N_γ as

$$\tan\gamma = -\frac{c_1^1 - c_3^1}{s_1^1 - s_3^1}\,. \tag{4.39}$$

Imposing the equality of the t-components of v_1' and v_3', and using the value of γ found above, leads to the condition

$$(c_1^0 - c_3^0)[(s_1^1 - s_3^1)^2 - (c_1^1 - c_3^1)^2] - 2(s_1^0 - s_3^0)(c_1^1 - c_3^1)(s_1^1 - s_3^1) - z_3^0[(s_1^1 - s_3^1)^2 + (c_1^1 - c_3^1)^2] = 0\,, \tag{4.40}$$

which is an algebraic equation for b_4. Solving this equation, and substituting the value of b_4 in (4.39), leaves as free parameters ℓ, R, ν and λ, which are the parameters one expects for a black a ring in Taub-NUT with one independent angular momentum: ℓ sets the scale of the KK circle, R is a measure of the radius of the ring, ν is a measure of the ratio between the radii of S^2 and S^1 at the horizon, and λ controls the angular momentum in the plane of the ring.

4.4 Balanced rings

In order to avoid a conical singularity at the location of a given rod, the period Δ_i of the spacelike coordinate ϕ_i(= a linear combination of ϕ_+, ϕ_-) vanishing there must be fixed as

$$\Delta_i = 2\pi \lim_{\rho\to 0} \sqrt{\frac{\rho^2 g'_{\rho\rho}}{|v_i'|^2}} \qquad \text{for} \qquad z \in I_i\,, \tag{4.41}$$

where $g'_{\rho\rho}$ is the $\rho\rho$ component of the metric determined by the Maison data (4.35), and $|v'_i|$ is the norm of v'_i. We find,

$$\Delta_1 = \Delta_4 = 2\pi\, \frac{k\left|b_4(1+\lambda)(1-\nu)^2 + (1-\lambda)(1+\nu)^2\right|}{(1-\lambda)(1+\nu)^2}\,, \tag{4.42a}$$

$$\Delta_3 = 2\pi\, \frac{k\,|1+b_4|(1-\nu)}{1+\nu}\sqrt{\frac{1+\lambda}{1-\lambda}}\,. \tag{4.42b}$$

Since for a ring one has $v'_1 = v'_3$, one needs to impose $\Delta_1 = \Delta_3$ in order to cancel the possible conical singularities. This condition can be solved for λ, and one finds that there are two solutions

$$\lambda = \frac{2\nu}{1+\nu^2}\,, \tag{4.43}$$

$$\lambda = \frac{(1+\nu)^2 - b_4^2(1-\nu)^2}{(1+\nu)^2 + b_4^2(1-\nu)^2}\,. \tag{4.44}$$

It can be checked that for the value of λ given in (4.44) the rod structure of the solution degenerates: the eigenvector of the semi-infinite rod, v'_1, becomes parallel to the eigenvector of the other semi-infinite rod, v'_4. This is the rod structure of a solution which has $\mathbb{R}^{3,1}\times S^1$ asymptotics, rather than Taub-NUT, and we can thus discard this solution. The requirement of absence of conical singularities, then uniquely fixes λ to take the value (4.43), which is the same value one finds for the S^1-rotating black ring in flat space. Notice this coincides with the perturbative analysis of section 3. This is the case we will consider in the following.

Restricting to balanced rings also simplifies the expressions considerably. Choosing,[11]

$$k = \frac{2}{|1+b_4|} \tag{4.45}$$

the periodicities of the angular directions become

$$\Delta\phi_- = \Delta\phi_+ = 4\pi\,, \tag{4.46}$$

and the value of the angle γ is then given by

$$\tan\gamma = \hat{R}\,\nu\sqrt{\frac{1+\nu}{1-\nu}}\,\frac{2(1-\nu)+\hat{R}^2 - b_4(2(1-\nu)-\hat{R}^2)}{(1-\nu)+\hat{R}^2(1+2\nu-\nu^2)+b_4\hat{R}^2(1+\nu)^2}\,. \tag{4.47}$$

As for the parameter b_4, one finds that equation (4.40) admits the following three solutions:

$$b_4^{(1)} = -\frac{4(1-\nu)+\hat{R}^2(1+\nu^2)}{\hat{R}^2(1-\nu^2)}\,, \tag{4.48}$$

$$b_4^{(2)} = \frac{1}{\hat{R}^2(4-\hat{R}^2)(1-\nu^2)^2}\Big[2\sqrt{(2(1-\nu)+\hat{R}^2\nu)^3(2(1-\nu)-\hat{R}^2\nu(1-2\nu^2))} \\ -8(1-\nu)^2 - 4\hat{R}^2\nu(1-\nu)(1+\nu^2) + \hat{R}^4(1-2\nu^2-\nu^4)\Big]\,, \tag{4.49}$$

$$b_4^{(3)} = \frac{1}{\hat{R}^2(4-\hat{R}^2)(1-\nu^2)^2}\Big[-2\sqrt{(2(1-\nu)+\hat{R}^2\nu)^3(2(1-\nu)-\hat{R}^2\nu(1-2\nu^2))} \\ -8(1-\nu)^2 - 4\hat{R}^2\nu(1-\nu)(1+\nu^2) + \hat{R}^4(1-2\nu^2-\nu^4)\Big]\,. \tag{4.50}$$

[11] This guarantees that the four-dimensional radial coordinate is canonically normalized.

Out these, only $b_4^{(2)}$ leads to a regular black ring. One finds, indeed, that the metric corresponding to $b_4^{(1)}$ has a degenerate rod structure. The rod 2 has the same direction as its two neighboring (spacelike) rods and the full rod structure is identical to that of flat space. Moreover the metric has singularities at the end-points of rod 2. For the solution $b_4^{(3)}$, we have checked numerically that the corresponding final metric has both naked singularities and CTCs outside the horizon, and it is thus physically unacceptable. This leaves us with only the solution $b_4^{(2)}$. One can verify that the corresponding metric is regular and free of CTCs, and, moreover, reduces to the S^1-rotating black ring in flat space when the radius of the KK direction becomes much larger than the scale of the ring. Therefore, from now on we will only consider the solution with $b_4 = b_4^{(2)}$.

4.5 Final solution

At this stage, our solution for the black ring in Taub-NUT is specified by $b_4^{(2)}$, γ and λ given in equations (4.49), (4.47) and (4.43) respectively. The geometry corresponding to the data (4.35) has a horizon with topology $S^2 \times S^1$ and no conical singularities. However, the solution has Dirac-Misner strings, which can be seen from the fact that the directions of the spacelike rods have compoeenents along ∂_t. To cure this pathology one still needs to apply a transformation $N_\alpha \in SO(2,1)$: this leads to a metric specified by the following Maison data

$$\chi'' = N_\alpha \, \chi' \, N_\alpha^T \,, \quad \kappa'' = (N_\alpha^{-1})^T \, \kappa' \, N_\alpha^T \,, \tag{4.51}$$

where

$$N_\alpha = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 \\ \sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.52}$$

The value of α needed to cancel the Dirac-Misner strings is found to be

$$\tanh\alpha = \frac{\hat{R}\,\nu(1-\nu)D_\nu D_2 - (1+\nu)^2 D_1\, s_{2\gamma}}{-4(1+\nu)^2 D_3\, c_\gamma + 2\,\hat{R}\,\nu(1-\nu)D_\nu\big[D_2 + 8\, b_4(1-\nu)\big]\, s_\gamma} \,, \tag{4.53}$$

where $s_\gamma \equiv \sin\gamma$, $c_\gamma \equiv \cos\gamma$ and the constants D_ν, D_1, D_2 and D_3 are defined in appendix D. Therefore, our black ring in Taub-NUT is given by the Maison data (4.51), with the parameters λ, γ, $b_4^{(2)}$ and α fixed as in equations (4.43), (4.49), (4.47) and (4.53) respectively. This completes our construction of the black ring in Taub-NUT.

At this point it is worth summarizing the main steps in our construction:

1. Start form the seed metric (4.24) and construct the corresponding Maison data χ and κ, as shown in the appendix C.

2. Apply the $SL(3,\mathbb{R})$ transformation (4.35).

3. Fix the parameters λ, b_4 and γ to the values given in (4.43), (4.49) and (4.47).

4. Apply the transformation (4.51), with α given in (4.53);

5. Reconstruct the metric from χ'' and κ''.

The final solution can be written in the form:

$$
\begin{aligned}
ds^2 = \lambda'_{11}\big(d\xi^1 + \mathbf{A}\big)^2 - \frac{\tau'}{\lambda'_{11}}\big(dt + \mathbf{B}\big)^2 \\
+ \frac{1}{\tau'}\,\frac{\ell^2 \hat{R}^4}{(x-y)^4}\bigg\{ \frac{H(x,y)}{(1+b_4^2)H(y,x)}\,\big[F(x,y) - F(y,x) + 2\,J(x,y)\big]\bigg(-\frac{dy^2}{G(y)} + \frac{dx^2}{G(x)}\bigg) \\
- G(x)G(y)\,d\phi_-^2 \bigg\}\,,
\end{aligned}
\tag{4.54}
$$

where

$$
\mathbf{A} = \frac{\lambda'_{01}}{\lambda'_{11}}\,dt + 2\,\frac{\lambda'_{01}\,\omega'^{0}_{-} + \lambda'_{11}\,\omega'^{1}_{-}}{\lambda'_{11}}\,d\phi_-\,, \qquad \mathbf{B} = 2\,\omega'^{0}_{-}\,d\phi_-\,, \tag{4.55}
$$

and the primed functions are obtained after the sequence of transformations described in the previous section. These can be read off from (4.51), which is given in terms of the χ-matrix of the seed solution (4.24) (see appendix C). In (4.54) and (4.55) we have rescaled the ϕ_- angle, $\phi_- \to 2\,\phi_-$, so that $\Delta\phi_- = 2\pi$.[12] To obtain a regular (five-dimensional) solution, the parameters in the metric (4.54) should obey

$$
0 < \nu < 1\,, \qquad 0 < \hat{R} < \sqrt{2}\,. \tag{4.56}
$$

5 Exact extremal ring in Taub-NUT

An appropriate extremal limit of the solution found in the previous section gives an exact geometry representing a D0-D6 system at zero temperature, with rotation only along the direction of the ring. The solution has a singular horizon of vanishing area. By studying this geometry for generic values of the distance between D0 and D6 charges, we compute their exact interaction potential.

5.1 The solution

In the parametrization of section 4, the extremal solution is obtained by taking $\nu = 0$: in this limit the horizon degenerates to a singular point. The solution one obtains has only one independent angular momentum, corresponding to rotation along the ring direction. As in the case of black rings in flat space, this S^1-rotating extremal solutions cannot be balanced: when $\nu = 0$ the condition of absence of conical singularities, eq. (4.43), has only the trivial solution $\lambda = 0$. Hence the space-time has a conical singularity, localized along the rod $\rho = 0$, $z \in [0, R^2/2]$ (or $x = 1$).

When $\nu = 0$, the construction of section 4 drastically simplifies. In particular, eq. (4.40), which guarantees that the t components of rods 1 and 3 be aligned, is satisfied for any value of b_4. Thus the parameter b_4 remains unfixed, and one has, in principle, a valid extremal black ring for any value of b_4. It turns out, however, that solutions with different values of b_4 are just different parametrizations of the same physical solution. The solution with $b_4 = 0$ gives the simplest parametrization, and it is the one that we will consider in the following. In this case,

[12]The reason for doing this is that upon KK reduction along ξ^1, the angular coordinate ϕ_- becomes the four-dimensional azimuthal direction.

the value of γ, given by eq. (4.39), simplifies to

$$\tan\gamma = \frac{\hat{R}}{2\sqrt{2}}\sqrt{\frac{1+\lambda}{1-\lambda}}\,. \tag{5.1}$$

λ remains a free parameter of the solution, essentially determining its angular momentum. The boost paremeter α, needed to cancel the Dirac-Misner string, is fixed to be

$$\tanh\alpha = \hat{R}\,\frac{4(1-\lambda)+\lambda\hat{R}^2}{4(1-\lambda)-\lambda\hat{R}^2}\sqrt{\frac{1+\lambda}{8(1-\lambda)+\hat{R}^2(1+\lambda)}}\,. \tag{5.2}$$

Reconstructing the metric from the Maison data χ'' and κ'', and performing some trivial rescaling of coordinates, one can write the metric of the extremal black ring in Taub-NUT in the form

$$ds^2 = g_{11}(d\xi^1 + A^1_0 dt + A^1_- d\phi_-)^2 + g_{11}^{-1/2}\Big[-V(dt + A^0_- d\phi_-)^2 + V^{-1}(e^{2K}ds_B^2 + \hat{\rho}^2 d\phi_-^2)\Big]\,. \tag{5.3}$$

The part of the metric in square brackets is the 4D Einstein-frame metric. The metric depends on the parameters ℓ and R, which have dimensions of length, and on the dimensionless parameter λ. As before, we denote $\hat{R} = R/\ell$.

To write the metric coefficients in compact form, it is convenient to define the dimensionless constants

$$\mathcal{C}_1 = \big[4(1-\lambda)+\lambda\hat{R}^2\big]^2\,,\quad \mathcal{C}_2 = \big[4(1-\lambda)+\hat{R}^2\big]^2\,, \tag{5.4}$$

and the functions

$$\begin{aligned}
F_0 &= x + y + \lambda(1+xy)\\
F_1 &= \lambda\mathcal{C}_2\big[(1-\lambda)(1-x)^2 - (1+\lambda)(1+y)^2\big] + \mathcal{C}_1(1+\lambda)F_0\\
F_2 &= \mathcal{C}_2 F_0 - \mathcal{C}_1\big[\lambda(1+x)^2 + 2(1-\lambda)x\big]\\
F_3 &= \lambda\mathcal{C}_2\big[(1-\lambda)\big(x^2(x+y)-2x\big) + (1+\lambda)\big(y^2(x-y)+2(1-y^2)\big) + 2\lambda(-2+x-y+2xy)\big]\\
&\quad - \mathcal{C}_1(1+\lambda)\big[\lambda(x-y)(1+xy) - (1-2\lambda)x^2 - y^2 + 2(1-\lambda)\big]\,.
\end{aligned} \tag{5.5}$$

Then the metric coefficients are given by

$$\begin{aligned}
g_{11} &= \frac{(\mathcal{C}_2-\mathcal{C}_1)F_1}{\big[(1+\lambda)\mathcal{C}_1 - 2\lambda\mathcal{C}_2\big]F_2}\,,\\
V &= -\sqrt{\big[(1+\lambda)\mathcal{C}_1 - 2\lambda\mathcal{C}_2\big](\mathcal{C}_2-\mathcal{C}_1)}\,\frac{F_0}{(F_1F_2)^{1/2}}\,,\\
A^1_0 &= \lambda\sqrt{\frac{\mathcal{C}_2(1-\lambda^2)}{\mathcal{C}_2-\mathcal{C}_1}}\,\frac{(x-y)\big[\mathcal{C}_1(1-x)+\mathcal{C}_2(x+y)\big]}{F_1}\,,\\
A^1_- &= \frac{\ell\,\hat{R}^2}{2(\mathcal{C}_2-\mathcal{C}_1)}\sqrt{\frac{\mathcal{C}_1[(1+\lambda)\mathcal{C}_1 - 2\lambda\mathcal{C}_2]}{1-\lambda}}\,\frac{F_3}{(x-y)F_1}\,,\\
A^0_- &= \ell\,\hat{R}^2\sqrt{\frac{(1+\lambda)\mathcal{C}_1\mathcal{C}_2}{[(1+\lambda)\mathcal{C}_1 - 2\lambda\mathcal{C}_2](\mathcal{C}_2-\mathcal{C}_1)}}\,\frac{\lambda(1-x^2)(1+y)}{(x-y)F_0}\,,\\
e^{2K} &= \frac{F_0}{(1-\lambda)(x+y)}\,,
\end{aligned}$$

$$
\begin{aligned}
\hat{\rho}^2 &= \ell^2\, \hat{R}^4\, \frac{(1-x^2)(y^2-1)}{(x-y)^4}\,, \\
ds_B^2 &= -\ell^2\, \hat{R}^4\, \frac{(x+y)}{(x-y)^3} \left(\frac{dx^2}{1-x^2} + \frac{dy^2}{y^2-1} \right) .
\end{aligned} \tag{5.6}
$$

5.2 Physical parameters

The mass, charges and angular momentum of the solution are

$$
\begin{aligned}
M_{\rm tot} &= \frac{\ell\,\hat{R}^2}{8\,G_4}\, \frac{\mathcal{C}_1^2(1-\lambda)+2\lambda(\mathcal{C}_2-\mathcal{C}_1)^2}{(\mathcal{C}_2-\mathcal{C}_1)[(1+\lambda)\mathcal{C}_1-2\lambda\mathcal{C}_2]}\,, \\
P &= \frac{\ell\,\hat{R}^2}{4(\mathcal{C}_2-\mathcal{C}_1)} \sqrt{\frac{\mathcal{C}_1[(1+\lambda)\mathcal{C}_1-2\lambda\mathcal{C}_2]}{1-\lambda}}\,, \\
Q &= \frac{\ell\,\hat{R}^2\lambda}{2\big[(1+\lambda)\mathcal{C}_1-2\lambda\mathcal{C}_2\big]} \sqrt{\frac{(1+\lambda)\mathcal{C}_2(\mathcal{C}_2-\mathcal{C}_1)}{1-\lambda}}\,, \\
J &= \frac{\ell^2\hat{R}^4}{8\,G_4}\, \frac{\lambda}{1-\lambda} \sqrt{\frac{(1+\lambda)\mathcal{C}_1\mathcal{C}_2}{(\mathcal{C}_2-\mathcal{C}_1)[(1+\lambda)\mathcal{C}_1-2\lambda\mathcal{C}_2]}}\,.
\end{aligned} \tag{5.7}
$$

($M_{\rm tot}$ is the total mass, and thus includes the magnetic monopole contribution. This is different from M in (2.14).) Note that, as $\mathcal{C}_2 > \mathcal{C}_1 > 0$ and $0 \le \lambda < 1$, the combination $(1+\lambda)\mathcal{C}_1 - 2\lambda\mathcal{C}_2$ can become negative for some value of the parameters. This happens for $\hat{R}^2 > \hat{R}_0^2$, where

$$
\hat{R}_0^2 = \frac{4(1-\lambda)}{2+\lambda} \left[-1 + \sqrt{\frac{2(1+\lambda)}{\lambda}} \right] . \tag{5.8}
$$

Beyond this point the magnetic charge becomes imaginary, and the metric ceases to make sense. Thus the parameters have to be taken in the range

$$
\ell > 0\,, \quad 0 \le \hat{R} < \hat{R}_0. \tag{5.9}
$$

For this range of parameters the magnetic and electric charges P and Q attain all values from zero to $+\infty$, and thus the range (5.9) covers the whole physical spectrum of the charges.

Note however that the solution (5.3) only spans a codimension one subspace of the full phase space of extremal black rings in Taub-NUT, describing rings for which the angular momentum is linked to the charges as

$$
PQ = G_4 J\,. \tag{5.10}
$$

The condition above restricts to configurations that, when uplifted to 5D, have angular momentum in only one plane, which turns out to be the plane of the ring.

5.3 Limits

In this subsection we analyze the various limits that connect the metric (5.3) to previously known solutions. Taking the radius of the ring to zero one reproduces the extremal KK black hole with $G_4 J = PQ$ found in [20, 21]. We also show how to recover the extremal S^1–spinning ring in flat space found in [5]. This limit corresponds to localising the black ring near the tip of Taub-NUT space and zooming into that region.

Extremal KK black hole with $G_4 J = PQ$

We can recover the extremal KK black hole of [20, 21] with $G_4 J = PQ$ as the $R \to 0$ limit of our extremal black ring in Taub-NUT. One should keep in mind that the KK black hole with $G_4 J = PQ$ has zero area and, strictly speaking, should be regarded as a naked singularity.

Recall that from five-dimensional viewpoint, KK black holes with non-zero magnetic charge can be thought of black holes sitting at the tip of the Taub-NUT space. Therefore, this limit of our solution can be regarded as a zero-radius limit in which effectively we are localizing the black ring at the tip of the Taub-NUT space while keeping the radius of the Taub-NUT circle, and hence the magnetic charge P, as well as the electric charge Q, fixed. This is achieved by taking $R \to 0$ and $\lambda \to 1$ keeping fixed the parameters $\tilde{R}$ and ℓ, with

$$\tilde{R} = \frac{R}{\sqrt{1-\lambda}} \,. \tag{5.11}$$

One should also change coordinates $(x, y) \to (r, \theta)$ as

$$x \to -1 + \frac{R^2}{\ell r} \cos^2\left(\tfrac{\theta}{2}\right) , \qquad y \to -1 - \frac{R^2}{\ell r} \sin^2\left(\tfrac{\theta}{2}\right) . \tag{5.12}$$

The resulting metric is

$$ds^2 = \frac{H_q}{H_p}\left(d\xi^1 + \mathbf{A}\right)^2 - \frac{r^2}{H_q}\left(dt + \mathbf{B}\right)^2 + H_p\left(\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta\, d\phi^2\right) , \tag{5.13}$$

where

$$H_p = r^2 + r\,p + \frac{p^2 q}{2(p+q)}\left(1+\cos\theta\right) , \qquad H_q = r^2 + r\,q + \frac{p\,q^2}{2(p+q)}\left(1-\cos\theta\right) , \tag{5.14a}$$

$$\mathbf{A} = -\frac{1}{H_q}\left\{ Q\left[2r + p(1-\cos\theta)\right]dt + P\left[2H_q\cos\theta - q\left(r + \tfrac{p\,q}{p+q}\right)\sin^2\theta\right]d\phi \right\} , \tag{5.14b}$$

$$\mathbf{B} = \frac{(p\,q)^{3/2}}{2(p+q)r^2}\,\sin^2\theta\, d\phi \,. \tag{5.14c}$$

The parameters p and q (with $p > 0$, $q > 0$) are related to $\tilde{R}$ and ℓ as

$$\tilde{R} = \frac{p\sqrt{q(4p+3q)}}{2(p+q)} , \qquad \ell = \frac{p(4p+3q)}{4(p+q)} , \tag{5.15}$$

and the electric and magnetic charges are given by

$$Q^2 = \frac{q^3}{4(p+q)} , \qquad P^2 = \frac{p^3}{4(p+q)} \,. \tag{5.16}$$

The metric (5.13) is just the $a = m = 0$ KK black hole in the form presented in [21]. Moreover, one can check that in this limit the charges satisfy

$$2G_4 M_{\rm tot} = \left[Q^{2/3} + P^{2/3}\right]^{3/2} . \tag{5.17}$$

Extremal S^1–spinning black ring in flat space

In the limit in which the KK radius becomes much larger than the ring size, one expects to recover the extremal S^1–spinning black ring in flat space . This limit is achieved by sending the magnetic charge P to infinity while keeping finite the size of the black ring, which is roughly given by R. In our parametrization, we have to send $\ell \to \infty$ while R, λ and the coordinates (x, y) are kept fixed. To recover the extremal limit of the S^1–spinning ring of [5] in the form presented in [37], one also has to redefine the R parameter as $R \to R\sqrt{\frac{1-\lambda}{2}}$ and change the angular coordinates as

$$\phi_+ = \frac{1}{\sqrt{1-\lambda}}(\phi' - \psi') \,, \qquad \phi_- = \frac{1}{\sqrt{1-\lambda}}(\phi' + \psi') \,, \tag{5.18}$$

since the angular coordinates (ϕ', ψ') of [37] are not canonically normalized. Then we obtain:

$$\begin{aligned} ds^2 =& -\frac{F(y)}{F(x)}\left(dt - C\,R\,\frac{1+y}{F(y)}\,d\psi'\right)^2 \\ &+ \frac{R^2}{(x-y)^2}\,F(x)\left[-\frac{G(y)}{F(y)}\,d\psi'^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}\,d\phi'^2\right] \,, \end{aligned} \tag{5.19}$$

where $C = \lambda\sqrt{\frac{1+\lambda}{1-\lambda}}$, $F(\xi) = 1 + \lambda\xi$ and $G(\xi) = 1 - \xi^2$. This is the $\nu \to 0$ limit of the metric given in [37].

5.4 Interaction energy

A measure of the interaction energy between the D0 and D6 charges is given by

$$E_{\rm int} = M_{\rm tot} - (M_{\rm D0} + M_{\rm D6}) = M_{\rm tot} - \frac{Q+P}{2G_4} \,, \tag{5.20}$$

where $M_{\rm tot}$ is the total mass of the system, measured at infinity as the ADM mass, and $M_{\rm D0}$, $M_{\rm D6}$ are the masses of the D0 and D6 branes in isolation.

To obtain a physical understanding of $E_{\rm int}$ one should express it in terms of the charges Q and P and of the distance between them. A rough estimate of this distance is given by the parameter R. A more precise measure of distance, at least in the limit of large separation, when the distortion on the metric due to the interaction between D0 and D6 is small, is furnished by the length of the rod at $\hat\rho = 0$ and $0 < \hat z < R^2/(2\ell)$, computed with the 5D metric (5.3):

$$R_{\rm ph} = \int_{-\infty}^{-1} dy \sqrt{G^{(5)}_{yy}\Big|_{x\to 1}} \,, \tag{5.21}$$

where $G^{(5)}_{yy}$ is the yy component of the 5D metric (5.3):

$$G^{(5)}_{yy} = \frac{R^4}{\ell^2} g_{11}^{-1/2} V^{-1} e^{2K} \frac{(x+y)}{(x-y)^3(1-y^2)} . \tag{5.22}$$

In the limit $R \to 0$, with fixed Q and P, one finds that $R_{\rm ph}$ so defined goes to a non-zero value, given by

$$R_{\rm min} = 4 Q^{1/3} P^{2/3} \,. \tag{5.23}$$

This is a quite counterintuitive result: as we have shown in section 5.3, the black ring reduces in this limit to the extremal KK black hole, which, naively, represents the configuration in which the D0 and D6 charges are on top of each other. The fact that one finds instead a non-zero distance $R_{\rm min}$ can be attributed to the large distortion on the metric due to the D0-D6 interaction (we will show that in this limit the metric has the maximal conical defect angle $\Delta = 2\pi$). To correct for this effect, we redefine $R_{\rm ph}$ as $R_{\rm ph} \to R_{\rm ph} - R_{\rm min}$.

The behavior of $E_{\rm int}$ as a function of R, for different values of the ratio Q/P, was shown in fig. 1. $E_{\rm int}$ is a monotonically decreasing function of the distance, that goes to a positive value for $R = 0$ and vanishes for large R. Hence the interaction between D0 and D6 branes is always repulsive. The two limits of R small and large R can be understood analytically.

The limit of small R and fixed charges is the same as the limit in which the black ring reduces to the KK black hole. As the mass of the KK black hole is given by (5.17), the interaction energy in this limit is given by

$$2G_4 E_{\rm int} \approx (Q^{2/3} + P^{2/3})^{3/2} - (Q + P) > 0\,. \tag{5.24}$$

Keeping the terms of higher order in R, one can see that $E_{\rm int}$ has an extremum at $R = 0$ as a function of R. On the other hand, when expressed in terms of $R_{\rm ph}$, $E_{\rm int}$ has a non-vanishing slope at $R_{\rm ph} = 0$: this is due to the fact that the physical distance $R_{\rm ph}$ depends quadratically on R, for small R. However, as we have already noted above, $R_{\rm ph}$ does not seem to provide a good measure of the distance between D0 and D6 in this limit.

The opposite limit is when $R \gg Q$, P. To achieve this limit one should take R/ℓ large; note however that this cannot be done at fixed λ, due to the constraint (5.9). One should also adjust λ in such a way that the upper bound for R/ℓ, $\hat{R}_0$, becomes large, which happens if λ goes to zero. Thus the appropriate limit is

$$R = \frac{\ell}{\sqrt{\epsilon}}\,, \quad \lambda = \lambda_0 \epsilon^4\,, \quad \epsilon \to 0\,. \tag{5.25}$$

In this limit one has

$$\frac{Q}{P} \approx \frac{\lambda_0}{128}\,, \quad P \approx 4\epsilon\ell\,, \quad G_4 M_{\rm tot} \approx 2\ell\Big(1 + \frac{\lambda_0}{128}\Big)\epsilon + \frac{\ell}{4}\lambda_0\epsilon^3\,. \tag{5.26}$$

These relations imply that the interaction potential is given by

$$G_4 E_{\rm int} = 2^{1/3} Q \Big(\frac{P}{R}\Big)^{4/3}\,. \tag{5.27}$$

We can re-express this result in terms of the physical distance $R_{\rm ph}$, which, for large separations,[13] is given by

$$R_{\rm ph} \approx 2^{-1/3} P \Big(\frac{R}{P}\Big)^{4/3}\,. \tag{5.28}$$

Substituting this $R_{\rm ph}$ in the above expression for $E_{\rm int}$, one finds

$$G_4 E_{\rm int} = \frac{QP}{R_{\rm ph}}\,, \tag{5.29}$$

which has the form of a repulsive Coulomb potential between charges Q and P. This result is in agreement with the interaction energy (3.43) derived by the perturbative method.

[13] At large distances the five-dimensional physical distance coincides with the four-dimensional one (computed in Einstein frame) to leading order.

5.5 Conical defect

Further information on the interaction between the D0 and D6 charges are obtained by analyzing the conical defect singularity of the extremal solution (5.3). This singularity is what is needed to counter-balance the repulsion between the D0's and the D6's, and thus the stress tensor associated to it gives a measure of the interaction.

Following a method used in [39] in an analogous context, we will compute the delta-function-like contribution to the Ricci tensor at the conical singularity, and use Einstein's equation to derive the associated stress tensor.

To obtain a duality invariant description of the system, we focus on the 4D Einstein frame metric, given by

$$ds_4^2 = -V(dt + A^0 d\phi_-)^2 + V^{-1}\Big[e^{2K}(d\hat{\rho}^2 + d\hat{z}^2) + \hat{\rho}^2 d\phi_-^2\Big]\,, \tag{5.30}$$

where $\hat{\rho}$ and $\hat{z}$ are the 4D Weyl coordinates

$$\hat{\rho} = \frac{\rho}{\ell}\,, \quad \hat{z} = \frac{z}{\ell}\,. \tag{5.31}$$

In the vicinity of the rod $\hat{\rho} = 0$ and $0 < \hat{z} < R^2/(2\ell)$, one has that

$$A^0 \approx 0\,, \quad e^{2K} \approx \frac{1+\lambda}{1-\lambda}\,. \tag{5.32}$$

Thus near this rod the metric in the $\hat{\rho}$-ϕ_- plane decouples from the remaining directions and becomes conformally equivalent to the flat metric in $\mathbb{R}^2$ after the change of coordinates

$$\hat{\phi} = \frac{\phi_-}{e^K} = \sqrt{\frac{1-\lambda}{1+\lambda}}\phi_-\,. \tag{5.33}$$

The condition that at asymptotic infinity the metric be flat fixes the periodicity of ϕ_- to be 2π. We thus see that along the rod under consideration there is a conical defect given by

$$\Delta = 2\pi\Big(1 - \frac{1}{e^K}\Big) = 2\pi\Big(1 - \sqrt{\frac{1-\lambda}{1+\lambda}}\Big)\,. \tag{5.34}$$

Note that for our geometries the deficit angle Δ is always positive.

The curvature due to this conical defect can be computed from the general relation [40]

$$\int_{\mathcal{M}} R = 2\Delta \mathcal{A}_\Sigma\,, \tag{5.35}$$

where R is the Ricci scalar, $\mathcal{M}$ is the full space-time manifold, $\mathcal{A}_\Sigma$ is the space-time area of the conical defect. In our case Σ is the rod $0 < \hat{z} < R^2/(2\ell)$ times time, and its area is $\mathcal{A}_\Sigma = \int dt d\hat{z}\, e^K$. Thus, eq. (5.35) implies

$$R = 2\Delta V \delta(\hat{\rho})\,, \tag{5.36}$$

where the delta-function $\delta(\hat{\rho})$ is normalized as $\int d\hat{\rho} d\phi_-\, \hat{\rho}\, e^K \delta(\hat{r}) = 1$. From the Ricci scalar R, and the fact that the curvature only has components along the $\hat{\rho}$-ϕ_- plane — so that

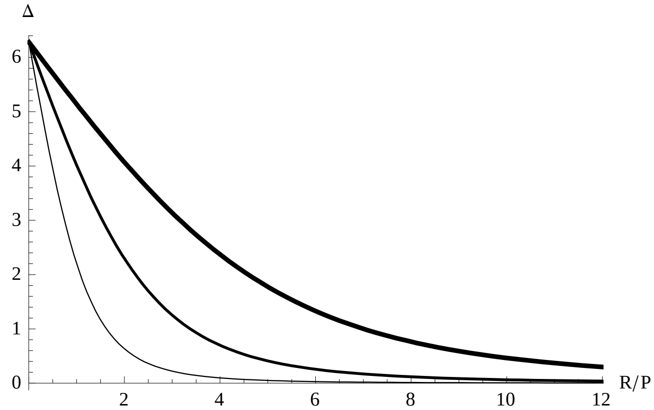

Figure 5: Conical defect versus radius R in units of P for $Q/P = .1$ (thin), 1 (thick) and 10 (thicker). Note that the conical defect is never zero for finite values of R. This means that the solution is never balanced. In the $R \to 0$ limit the conical defect is maximal: 2π. The force F_{def} is porportional to Δ (5.39).

$R_{00} = R_{zz} = 0$ — one can derive the Einstein tensor and hence the stress tensor $T_{\mu\nu}$ associated to the conical defect. One finds that

$$T_{00} = (8\pi G_4)^{-1} \Delta V^2 \delta(\hat{\rho}) \,, \quad T_{zz} = -(8\pi G_4)^{-1} \Delta e^{2K} \delta(\hat{\rho}) \,. \tag{5.37}$$

The energy of the conical defect, obtained by integrating the energy density $\frac{T_{00}}{\sqrt{-g_{00}}}$ over the space directions, is given by

$$E_{\text{def}} = \int d\hat{z} d\hat{r} d\phi_- \, \hat{\rho} \, e^{2K} \frac{T_{00}}{\sqrt{-g_{00}}} V^{-3/2} = (8G_4)^{-1} \left(\sqrt{\frac{1+\lambda}{1-\lambda}} - 1 \right) \frac{R^2}{\ell} . \tag{5.38}$$

The force exerted by the strut is obtained as the integral of the pressure $\frac{T_{zz}}{g_{zz}}$ over the directions transverse to the strut:

$$F_{\text{def}} = \int d\hat{\rho} d\phi_- \, \hat{\rho} \, e^{K} \frac{T_{zz}}{g_{zz}} V^{-1} = -(4G_4)^{-1} \left(1 - \sqrt{\frac{1-\lambda}{1+\lambda}} \right) . \tag{5.39}$$

We note that F_{def} is proportional to the conical defect Δ.

The plots describing the behavior of E_{def} and F_{def} as a function of the distance parameter R, for fixed values of the charges Q and P are depicted in figures 5 and 6.

One notes from these plots that the force is always repulsive and is maximal at $R = 0$. The energy E_{def} vanishes at large R, has a maximum at some finite value of R and vanishes again at $R = 0$. This seems to contradict the behavior found for the interaction potential E_{int}, which gave a repulsive potential for every value of R. We interpret the vanishing of E_{def} at small R as a volume effect: at $R = 0$ the metric has a maximal conical defect $\Delta = 2\pi$, and thus the space transverse to the strut becomes effectively one-dimensional, and its volume vanishes. Hence the fact that E_{def} vanishes does not mean that the D0-D6 interaction becomes attractive at small R.

The behavior of E_{def} and F_{def} in the limits of small and large separations can be reproduced

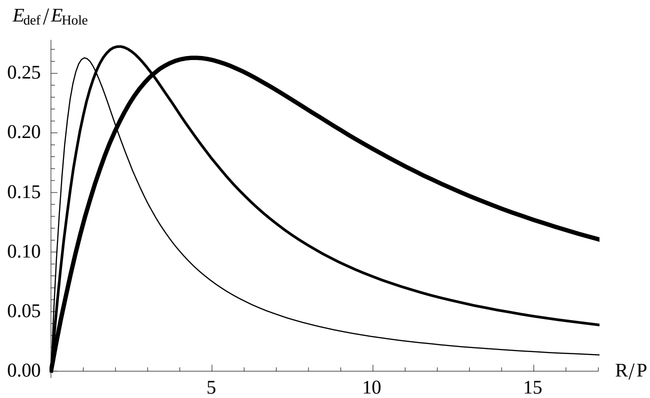

Figure 6: Energy of the conical defect versus R/P for $Q/P = .1$ (thin), 1 (thick) and 10 (thicker). Energies are normalised as in figure 1. The plot shows that $E_{\rm def}$ has a maximum at some finite value of R. The position of the maximum increases with Q/P. The vanishing of $E_{\rm def}$ at small R is a volume effect, and does not mean that the force between the charges becomes attractive at small distances.

analytically. For small R one finds

$$\begin{aligned} 8G_4 E_{\rm def} &\approx 2\sqrt{2} R \frac{Q^{1/3}}{\sqrt{4P^{2/3}+3Q^{2/3}}}\,, \\ 4G_4 F_{\rm def} &\approx -1 + \frac{R}{P^{2/3}Q^{1/3}} \sqrt{\frac{P^{2/3}+Q^{2/3}}{2(4P^{2/3}+3Q^{2/3})}}\,. \end{aligned} \tag{5.40}$$

The large R behavior is given by

$$\begin{aligned} 8G_4 E_{\rm def} &\approx 2^{7/3} Q \left(\frac{P}{R}\right)^{4/3}, \\ 4G_4 F_{\rm def} &\approx -2^{5/3} \frac{Q}{P}\left(\frac{P}{R}\right)^{8/3}. \end{aligned} \tag{5.41}$$

Once expressed in terms of the physical distance $R_{\rm ph}$, given in this limit in (5.28), these expressions simplify to

$$\begin{aligned} 2G_4 E_{\rm def} &\approx \frac{QP}{R_{\rm ph}}\,, \\ 2G_4 F_{\rm def} &\approx -\frac{QP}{R_{\rm ph}^2}\,, \end{aligned} \tag{5.42}$$

which are again of Coulombic form. Note that, in this limit, the energy of the conical defect accounts for half of the interaction energy between D0 and D6 branes: $E_{\rm def} = E_{\rm int}/2$.

6 Exact black rings in Taub-NUT

Using the exact solution constructed in section 4, we can study the D0-D6 system above extremality in a regime in which the gravitational backreaction of the D0 branes on the D6 Taub-NUT background is fully taken into account. We are interested in configurations of equilibrium, in which conical singularities have been eliminated.

In this analysis, one should keep in mind that the solution of section 4 has only one independent angular momentum, and thus it does not represent the most general black ring in Taub-NUT. Indeed, the starting seed solution (4.24) only has intrinsic rotation along the S^1 of the ring. This can be seen from the fact that only the BZ transformation affecting the horizon rod gives rotation to the solution. The angular momentum on the S^2 is induced by the soliton transformation on the spacelike rods, which mixes the $\hat{\phi}$ and $\hat{\psi}$ directions. It is precisely this lack of generality that implies that the four-dimensional conserved charges cannot all be independent. Recall that from a four-dimensional perspective, our solution is characterized by the four conserved charges ($M_{\rm tot}$, J, Q, P), see appendix D. However, our (balanced) solution only has three free parameters, namely, an overall length scale ℓ and two dimensionless quantites, $(\hat{R}, \nu)$. Therefore, once we have fixed this overall scale fixing, say, P, there must exist a relation between the remaining conserved charges, which relates the angular momentum J along the S^2 to Q, P and $M_{\rm tot}$. At the limiting endpoints of the family of solutions, *i.e.,* when the ring is infinitely far from the nut, or when it collapses into a singular extremal black hole, this relation becomes $J = PQ/G_4$, but at any other point in the space of our solutions we have $J \neq PQ/G_4$, signaling a non-zero component along the S^2 of the ring.

A more general doubly spinning black ring in Taub-NUT must exist with independent rotation along the S^2 which should allow to vary J independently of PQ, so the four dimensional solution would be characterized by four independent conserved charges. In fact such solutions are easily described in the thin ring limit within the perturbative approach, see appendix E. In this paper we shall content ourselves with studying in detail the physics of the black ring in Taub-NUT with just one independent angular momentum. We leave the problem of constructing the general solution for future work.

From the four-dimensional perspective, we have an electrically charged rotating black hole at $y = -\frac{1}{\nu}$, separated from the magnetic monopole, which sits at $(x = +1, y = -1)$ and which appears as a naked singularity, although the five-dimensional geometry is regular.

6.1 Dimensionless quantities

We characterize the solution in terms of four-dimensional magnitudes, since the solution is asymptotically flat in the non-compact four dimensions.

In Kaluza-Klein solutions it is natural to fix the length of the KK circle and measure all dimensionful quantities relative to it. Since the asymptotic length of this circle is measured by the magnetic charge as in (1.1), an essentially equivalent way of doing this is to define dimensionless quantities by dividing the physical magnitudes by suitable powers of P. We first fix one of the dimensionless parameters by fixing the total mass

$$\frac{G_4 M_{\rm tot}}{P} \equiv \mu + \frac{1}{2} \,. \tag{6.1}$$

Note that $\mu \geq 0$, with $\mu = 0$ when the mass equals the D6 brane mass

$$M_{\rm D6} = \frac{P}{2\, G_4} \,. \tag{6.2}$$

Then μ measures the energy above the D6 brane mass. This is convenient, since we regard the D6 brane as remaining unexcited, while the D0 brane (the black ring) is thermally excited.

We define other dimensionless conserved quantities as

$$a_{\rm H} = \frac{\mathcal{A}_4}{P^2} \,, \qquad j = \frac{G_4\, J}{P^2} \,, \qquad q = \frac{Q}{P} \,. \tag{6.3}$$

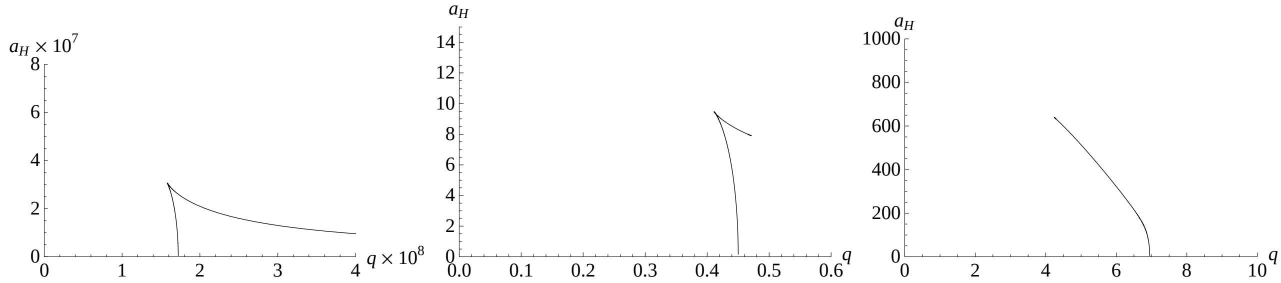

Figure 7: Phase diagram of black rings in Taub-NUT. *Left:* Phase diagram for $\mu = 5 \times 10^{-6}$. In this plot the value of the reduced area $a_{\rm H}$ and that of the reduced electric charge q have been multiplied by a factor of 10^7 and 10^8 respectively. *Center:* Phase diagram for $\mu = 1/2$. For small μ the curves resemble the ones in asymptotically flat five dimensions, but in contrast to them, the thin ring branch extends only up to a maximum value of the charge for given mass. The limiting values of the area and charge for fixed mass can be found from (3.34) and (3.37). *Right:* Phase diagram for $\mu = 9/2$. For $\mu > \mu_c = 9/10$ the thin ring branch disappears: we regard this as nothing more than a peculiar feature of the particular family of solutions we have constructed.

If we are interested in having a five-dimensional perspective on the solution, we note, using the relations (D.6), that the angular momenta J_ψ along the S^1 and J_ϕ along the S^2 of the black ring are captured by the five-dimensional dimensionless quantities

$$j_\psi = q \, , \qquad j_\phi = j - q \, , \tag{6.4}$$

see eq. (E.2). Thus q measures the S^1 spin of the ring. The four and five-dimensional horizon areas differ only in a constant factor since we are keeping L fixed and so a_H represents both.

6.2 Phase diagram

For fixed magnetic charge P, the area of the black ring is a two-dimensional surface over the plane of μ and q, $a_{\rm H}(\mu, q)$. In order to visualize it, we consider sections at constant μ. In figure 7 we present three illustrative plots, one at a very small value of μ, another at a value $\mu = 1/2$ that makes the D0 and D6 equally heavy, and a third one at very large D0 mass, $\mu = 9/2$.

Just like in the asymptotically flat case, black rings exist in two branches, usually referred to as thin and fat rings. Despite the similarity to the singly spinning ring in flat space [5], there is a small distorsion due to the fact that our black ring is doubly spinning, and the angular momentum in the S^2 of the ring varies along the curves shown in figure 8. Therefore it is more accurate to compare it to a family of doubly spinning black rings, whose S^2-spin is generically small and vanishes at the endpoint solutions.

For μ close to 0 the black ring can be thought of as a small perturbation in the KK monopole background. In this regime the phase diagram looks very similar to that of a single-spin black ring in flat space—observe in fig. 8 that for small μ we have $j \simeq q$ so $j_\phi \simeq 0$. However, the curves of a_H at fixed μ terminate at a finite value of the charge and area. The endpoints can in fact be precisely calculated: since they correspond to rings at infinite distance from the nut, they are accurately described by the construction of sec. 3. Thus the limiting values of the area and charge for fixed mass correspond to the critical values computed in (3.34) and (3.37). This is the case in fact for all values of μ, not just small ones. For small μ we can also find the

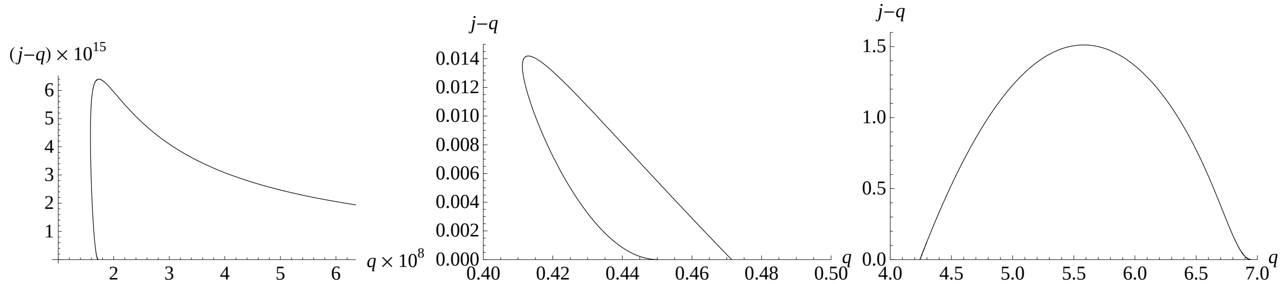

Figure 8: Phases in the $(q, j-q)$-plane. From left to right we show the phases corresponding to $\mu = 5\times 10^{-6}$, $1/2$ and $\mu = 9/2$ respectively. These curves represent only the subfamily constructed in this paper of a larger family of solutions with four-independent parameters, which should cover a finite region of the $(q, j-q)$-plane at every μ.

approximate values of the lower bound on q and upper bound on a_H using the bounds on black rings in flat space,

$$a_H \leq \frac{32\sqrt{2}\pi}{3\sqrt{3}}\mu^{3/2}\,, \qquad q \geq \sqrt{2}\,\mu^{3/2} \qquad (\mu \ll 1)\,. \tag{6.5}$$

These correspond in fact to the bounds quoted in (1.6).

As μ increases the lower limit on the area for fixed mass increases and the upper limit on the charge decreases (see (3.36)), so the thin ring branch shortens. As we move slightly away from this point the angular momentum along the S^2 of the ring switches on. The behavior of the phase diagram near the endpoint of the thin ring branch for arbitrary values of μ can be analytically derived from our exact solution: it is described by the limit in which the parameter ν is sent to zero, and $\hat{R}$ is taken of the form $\hat{R} = \sqrt{2}(1-\eta\nu)$, where η is a fixed parameter controlling the mass of the solution. In this limit the perturbative approximation of sec. 3 becomes accurate, and indeed it is possible to match the limit of our exact solution with a boosted Kerr string with fixed boost parameter and perturbative angular momentum (see appendix E).

As we keep increasing μ we observe that the thin ring branch disappears at the critical value of $\mu_c = 9/10$. We believe that this is an artifact of our (particular) solution and that a more general black ring in Taub-NUT should have two branches of solutions for all $\mu > 0$, in agreement with the perturbative construction of section 3, which clearly contains thin black rings far from the nut for every value of μ.

For all values of μ, there is a particular limiting value of q for which $a_{\mathrm{H}} \to 0$. This corresponds to the zero radius limit of our solutions, and we recover the extremal ($G_4 J = PQ$) nakedly singular KK black hole of [20, 21].

7 Outlook

Our study provides an example of how novel gravitational techniques can be applied to extract useful information about the dynamics of D-branes. In particular, we have uncovered a new way —via thermal excitation— to produce bound states of D0 and D6 branes, and exhibited how a certain extremal black hole can be formed by bringing together D0 and D6 branes in a continuous manner, with the black hole mass corresponding to the mass of the D0 and D6 constituents plus the interaction energy.

We have also used our methods to analyze the stability of the configurations. The reader may have noticed that we have only discussed stability with respect to changes in the distance between the D0 and D6 branes, which is to say, changes in the radial position of the black ring in the Taub-NUT background. Thin black rings are known to be stable to such changes in asymptotically flat space [29], but on the other hand, they are expected to be generically *unstable* to Gregory-Laflamme-type of modes that create inhomogeneities along the ring and that would be missed by our analysis. Should not we expect our black rings to suffer from them, too?

The answer is that such instabilities can in fact be avoided. Extremal singular rings, corresponding to zero-temperature D0 branes, are certainly not expected to suffer from them. Non-extremal black rings should suffer from GL instabilities only if they are thinner than the Kaluza-Klein radius, $r_0 < L$. However, we have seen that within the scope of our methods we can study black rings with $r_0 > L$ as long as they are far from the nut, $R \gg r_0$. Such black rings resemble black strings that are not afflicted by the GL instability. We have found that they are also radially stable so, since no other mechanism for instability is known to affect them, we can expect these black rings to be stable. While we have not discussed in any detail *fat* black rings (with, roughly, $r_0 \sim R$), for example those that correspond to the lower branches of the curves in fig. 7, the generic arguments of [29] lead us to expect them to be radially unstable.

There is a number of possible extensions of our work. For instance, the perturbative techniques can be easily extended to other backgrounds of Taub-NUT type, supersymmetric or otherwise, and also to charged black rings (see *e.g.,* [41]).

Perhaps more interestingly, with the present techniques for generating exact solutions we could also obtain solutions where a black hole sits at the nut. This would correspond to thermally exciting the D6 branes, and presumably it would allow for equilibrium states of extremal or non-extremal D0 branes a finite distance apart. In five dimensions, the configuration can be regarded as a black saturn [8] in Taub-NUT, a non-supersymmetric analogue of the solution in [16]. Upon Kaluza-Klein reduction it would describe an electric and a magnetic black hole, both of them generically rotating under the effect of the Poynting-induced angular momentum, and in equilibrium at a finite distance from each other. In fact, in principle it should be possible to introduce an *arbitrary number* of black rings, which from the four-dimensional viewpoint would yield a set of electric black holes plus a magnetic one, generically non-extremal, rotating and aligned along a common axis, and in dynamical (although not thermodynamical) equilibrium.

This would be, to our knowledge, the first known way of achieving equilibrium in an asymptotically flat, non-supersymmetric and non-extremal multi-black hole configuration in four dimensions. Such configurations should also be stable bound states of black holes. We leave this interesting problem for future work.

Acknowledgements

We thank Iosif Bena, Veronika Hubeny, James Lucietti, Mukund Rangamani, Simon Ross, and Hossein Yavartanoo for useful discussions. PF and JC thank CEA for hospitality. JC also thanks the Durham University and the Hebrew University of Jerusalem for hospitality. The work of JC and RE was supported in part by DURSI 2005 SGR 00082, MEC FPA-2007-66665-C02, and the European Community FP6 program MRTN-CT-2004-005104. JC is also supported by a MICINN FPU grant ref AP2005-3120. A.S. was supported in part by the Marie

Curie IRG 046430. PF is supported by STFC.

A Supergravity solution for D6 brane with B-flux

We want to construct a supersymmetric solution describing a D6 brane with B flux in its worldvolume. We shall consider the fluxes to be homogeneous and isotropic in the D6 worldvolume, which for simplicity we may consider to be wrapping a square T^6. In this case the solution we seek, when lifted to M-theory and then reduced along the T^6, is a solution to the minimal supergravity theory in five dimensions.

The required class of solutions has been discussed in [31, 32, 16, 17, 18] (we follow mostly [16]). The solutions have metric

$$ds^2 = -Z^{-2}(dt + \omega)^2 + Z h_{mn} dx^m dx^n \tag{A.1}$$

with h_{mn} the metric of a hyper-Kähler base space, and gauge potential

$$A = \frac{\sqrt{3}}{2}\left[Z^{-1}(dt + \omega) - \beta\right] . \tag{A.2}$$

We take the base space to be a single-center Gibbons-Hawking space,

$$h_{mn} dx^m dx^n = H^{-1}(d\psi + (\cos\theta - 1)d\phi)^2 + H(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \tag{A.3}$$

with

$$H = h + \frac{1}{r} . \tag{A.4}$$

The solution is fully specified in terms of three more harmonic functions H_p, H_q, H_0 in three-dimensional space, as [14]

$$Z = H_q + \frac{H_p^2}{H} \tag{A.5}$$

and

$$\omega = \omega_0(d\psi + (\cos\theta - 1)d\phi) + \tilde{\omega} , \qquad \beta = \beta_0(d\psi + (\cos\theta - 1)d\phi) + \tilde{\beta} . \tag{A.6}$$

with

$$\omega_0 = -\frac{H_0}{2} + \frac{3H_p H_q}{2H} + \frac{H_p^3}{H^2} , \qquad \beta_0 = \frac{H_p}{H} . \tag{A.7}$$

The equations that determine the one-forms $\tilde{\omega}$ and $\tilde{\beta}$ in terms of H, H_p, H_q, H_0 can be found in the references mentioned above.

The residues of poles in (H, H_p, H_q, H_0) are respectively associated to numbers of D6, D4, D2, D0 branes, so our choice for H in eq. (A.4) corresponds to having a single D6 brane at the origin. Indeed, when $H_p = 0, H_q = 1, H_0 = 0$ we recover the solution for a single KK monopole. We now want to introduce B-field moduli corresponding to D4 branes 'dissolved' in the worldvolume of the D6. These will also induce D0 and D2 charges, but we do not want to introduce 'pure' D0's and D2's. So we set H_p, H_q, H_0 to be constant moduli h_i, without any poles,

$$H_p = h_p , \qquad H_q = h_q , \qquad H_0 = h_0 . \tag{A.8}$$

[14]To compare to the notation in [16], change $(H, H_p, H_q, H_0) \to (H_k, K, L, -2M)$, and $Z \to H$.

With these values we easily find that $\tilde{\beta} = 0$, and that $\tilde{\omega} = \frac{h_0}{2}(\cos\theta - 1)d\phi$. This last term introduces pathological Dirac-Misner strings involving the time direction and so it must be set to zero. Then

$$h_0 = 0 \quad \text{and} \quad \tilde{\omega} = 0\,. \tag{A.9}$$

Let us now rewrite the solution in a manner convenient for KK reduction,

$$\begin{aligned} ds^2 &= \frac{\Sigma^2}{H^2 Z^2}\left(d\psi + (\cos\theta - 1)d\phi - \frac{\omega_0 H^2}{\Sigma^2}dt\right)^2 \\ &\quad + \frac{HZ}{\Sigma}\left(-\frac{1}{\Sigma}dt^2 + \Sigma(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2)\right)\,. \end{aligned} \tag{A.10}$$

where $\Sigma = \sqrt{Z^3 H - \omega_0^2 H^2}$ is the 'entropy function'. Since $r^2\Sigma$ vanishes at the D6 core at $r = 0$, there is no entropy associated to this configuration, as expected. The full five-dimensional solution is in fact smooth there.

We shall restrict the remaining moduli h, h_p, h_q by demanding that the solution asymptotes to the Kaluza-Klein monopole vacuum with asymptotic circle radius L. We demand that as $r \to \infty$

$$\Sigma \to \frac{L}{2}\,, \qquad HZ \to 1\,. \tag{A.11}$$

This imposes two relations among the moduli, namely,

$$\sqrt{3h_p^2 h_q^2 + 4hh_q^3} = L\,, \qquad h_p^2 + hh_q = 1\,, \tag{A.12}$$

which we solve in parametric form as in eq. (3.4).

B Exact supersymmetric D0-D6 bound states

It is not difficult to explicitly construct exact supergravity solutions for supersymmetric bound states of D0 and D6 branes. This was first done in an explicit manner in [31]. Such solutions only provide equilibrium configurations, with the distance between the branes fully fixed by the field $B \geq B_c$, so we cannot study how the interaction potential changes as B changes. Moreover, they do not provide any information about configurations in which supersymmetry is broken. Nevertheless, for completenes, and as a check on our approximate methods, we present here the configuration with one D6 brane and n_0 D0 branes. It takes the form of eqs. (A.1)-(A.7) but now

$$\begin{aligned} H &= \frac{2}{L}\frac{1 - 3b^2}{\sqrt{(1+b^2)^3 + 2bn_0(3-b^2) + n_0^2}} + \frac{1}{r}\,, \\ H_p &= \frac{2b(1+b^2) + n_0}{\sqrt{(1+b^2)^3 + 2bn_0(3-b^2) + n_0^2}}\,, \\ H_q &= \frac{L}{2}\frac{(1+b^2)^2 + 2bn_0}{\sqrt{(1+b^2)^3 + 2bn_0(3-b^2) + n_0^2}}\,, \\ H_0 &= \frac{L^3}{8}\left(\frac{2}{L}\frac{(1-3b^2)n_0}{\sqrt{(1+b^2)^3 + 2bn_0(3-b^2) + n_0^2}} + \frac{n_0}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}\right) \end{aligned} \tag{B.1}$$

and the distance between the D0 and D6 is fixed to $r = R$, with

$$R = \frac{L}{2} \frac{\sqrt{(1+b^2)^3 + 2bn_0(3-b^2) + n_0^2}}{3b^2 - 1} \,. \tag{B.2}$$

Observe that when $n_0 = 0$ we reproduce the same background as in the previous subsection. Moreover, the equilibrium distance (3.30), obtained in the limit where the backreaction from the D0 branes, agrees with the exact result (B.2) when $n_0 \to 0$.

C Maison data

In this appendix we present the computation of the Maison data (χ, κ) of our seed solution (4.24). To avoid cluttering formulas, we will set $\ell = 1$ and we will restore the units when needed.

As a first step, we compute the quantities λ_{ab}, τ and $\omega^a = \omega^a{}_- d\phi_-$, needed to rewrite the seed metric (4.24) in the form (4.3). This only requires algebraic manipulations, and one finds:

$$\lambda_{00} = -\frac{H(y,x)}{H(x,y)} \,, \tag{C.1a}$$

$$\lambda_{01} = -\frac{R\, C_1}{2H(x,y)} \left(\omega_\psi + \bar{b}_4\, C_2\, \omega_\phi\right) , \tag{C.1b}$$

$$\lambda_{11} = \frac{R^2}{4(x-y)^2} \frac{1}{H(y,x)} \left[F(x,y) - F(y,x) + 2J(x,y) - \frac{C_1^2 (x-y)^2}{H(x,y)} \left(\omega_\psi + \bar{b}_4\, C_2\, \omega_\phi\right)^2 \right] , \tag{C.1c}$$

$$\tau = \frac{R^2}{4(x-y)^2} \frac{1}{H(y,x)} \left[F(x,y) - F(y,x) + 2J(x,y) \right] , \tag{C.1d}$$

$$\omega^0{}_- = \frac{R\, C_1}{H(y,x)} \frac{\omega_\psi \left[F(x,y) + J(x,y) \right] + \bar{b}_4\, C_2\, \omega_\phi \left[F(y,x) - J(x,y) \right]}{F(x,y) - F(y,x) + 2J(x,y)} \,, \tag{C.1e}$$

$$\omega^1{}_- = \frac{F(x,y) + F(y,x)}{F(x,y) - F(y,x) + 2J(x,y)} \,. \tag{C.1f}$$

To compute the scalar potentials V_a and the matrix κ, one needs instead to solve differential equations, which is, in practice, a difficult task in our case. However it was shown in [12] that one can relate V_a and κ to some auxiliary matrices, Γ and $\tilde{\kappa}$, which can be computed via a generalization of BZ techniques.

The Γ and $\tilde{\kappa}$ matrices

The Γ_0 and $\tilde{\kappa}_0$ matrices of the solution $\tilde{G}_0$ are given by

$$\Gamma_0 = \frac{1}{2} \operatorname{diag} \{ \mu_3 - \bar{\mu}_1 - \bar{\mu}_4 \,, \bar{\mu}_2 - \bar{\mu}_1 \,, \bar{\mu}_3 - \bar{\mu}_2 \} \,, \tag{C.2}$$

$$\tilde{\kappa}_0 = \frac{1}{8} \operatorname{diag} \{ \mu_3^2 - \bar{\mu}_1^2 - \bar{\mu}_4^2 \,, \bar{\mu}_2^2 - \bar{\mu}_1^2 \,, \bar{\mu}_3^2 - \bar{\mu}_2^2 \} \,. \tag{C.3}$$

To construct the Γ and $\tilde{\kappa}$ matrices of the seed solution (4.18) one can use the fact that the metric G can be constructed in a two-step process,

$$G = \mu_4 \left(\mathbf{1} - \frac{\rho^2 + \mu_4^2}{\mu_4^2} \mathbf{P}_2 \right) \left(\mathbf{1} - \frac{\rho^2 + \bar{\mu}_1^2}{\bar{\mu}_1^2} \mathbf{P}_1 \right) \tilde{G}_0 \tag{C.4}$$

where $\mathbf{1}$ is the 3×3 identity matrix, and $\mathbf{P}_{1,2}$ are the projectors

$$(\mathbf{P}_i)_{ab} = \frac{m_c^{(i)}(G_0)_{ca}\, m_b^{(i)}}{m_d^{(i)}(G_0)_{df}\, m_f^{(i)}}\,. \tag{C.5}$$

In the equation above G_0 denotes the seed metric at each step and $m_a^{(i)}$ are the vectors constructed out of the BZ vectors $m_{0a}^{(i)}$ and the seed solution at each step,

$$m_a^{(i)} = m_{0b}^{(i)}\left[\Psi_0^{-1}(\mu_i,\rho,z)\right]_{ba}\,. \tag{C.6}$$

Defining a new matrix $\mathbf{Q}_i$ as

$$\mathbf{Q}_i = \frac{\rho^2+\mu_i^2}{\mu_i}\,\mathbf{P}_i\,, \tag{C.7}$$

one can show that the Γ and $\tilde{\kappa}$ matrices of the solution G are given by

$$\begin{aligned}\Gamma &= \Gamma_0 + \frac{1}{2}\sum_{i=1}^{2}\mathbf{Q}_i + \frac{1}{2}\,\bar{\mu}_4\mathbf{1}\,, &\text{(C.8)}\\ \tilde{\kappa} &= \tilde{\kappa}_0 + \frac{1}{8}\left(\frac{\bar{\mu}_1^2-\rho^2}{\bar{\mu}_1}\,\mathbf{Q}_1 + \frac{\mu_4^2-\rho^2}{\mu_4}\,\mathbf{Q}_2\right) + \frac{1}{4}\sum_{i=1}^{2}\left[\mathbf{Q}_i,\Gamma_0\right] - \frac{1}{8}\left[\mathbf{Q}_1,\mathbf{Q}_2\right] + \frac{1}{8}\,\bar{\mu}_4^2\,\mathbf{1}\,. &\text{(C.9)}\end{aligned}$$

The explicit expressions of the Γ and $\tilde{\kappa}$ matrices are too involved to be written down here.

V_a and κ

The potentials V_a can be computed from the matrix Γ derived above as

$$V_0 = \Gamma_0{}^- + c_0\,, \qquad V_1 = \Gamma_1{}^- + c_1\,, \tag{C.10}$$

where the subscripts $\pm$ denote components in the base (4.2), and $c_{0,1}$ are constants which are determined by the asmptotic boundary conditions. As discussed in [10, 12], the matrix χ, computed from these data, should approach a constant matrix η_5 at asymptotic spatial infinty,

$$\chi \xrightarrow{r\to\infty} \eta_5 \equiv \begin{pmatrix} -1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\,. \tag{C.11}$$

In the (x,y) coordinates that we are using, spatial infinity lies at $x\to y\to -1$, and the correct asymptotics (C.11) is achieved by taking

$$c_0 = 0\,, \qquad c_1 = \frac{R^2(\lambda-2\nu+\lambda\nu)}{4(1-\lambda)}\,. \tag{C.12}$$

One can also check that the sub-leading correction to the asymptotic limit of the χ matrix is of the form

$$\chi = \eta_5\left[\mathbf{1} - \frac{\delta\chi}{r^2} + O\left(\tfrac{1}{r^4}\right)\right]\,, \tag{C.13}$$

where $\delta\chi$ is a constant 3×3 matrix. The matrix $\delta\chi$ contains the information about the conserved charges of the solution.

The last piece of data needed is the matrix of one-forms κ. The components of this matrix can be derived from the following relations [12]:

$$\begin{aligned}
\kappa_{00} &= V_0\,\omega^0 + \Gamma_0{}^0 + c_{00}\,,\\
\kappa_{01} &= V_1\,\omega^0 + \Gamma_+{}^0 + c_{01}\,,\\
\kappa_{02} &= -\omega^0 + c_{02}\,,\\
\kappa_{10} &= V_0\,\omega^1 + \Gamma_0{}^+ + c_{10}\,,\\
\kappa_{11} &= V_1\,\omega^1 + \Gamma_+{}^+ + c_{11}\,,\\
\kappa_{12} &= -\omega^1 + c_{12}\,,\\
\kappa_{20} &= V_0\left(V_0\,\omega^0 + V_1\,\omega^1\right) + \frac{1}{2}\left(\Gamma\sigma\Gamma\right)_0{}^- + \tilde{\kappa}_0{}^- + c_0\left(\Gamma_0{}^0 - \Gamma_-{}^- - z\right) + c_1\,\Gamma_0{}^+ + c_{20}\,,\\
\kappa_{21} &= V_1\left(V_0\,\omega^0 + V_1\,\omega^1\right) + \frac{1}{2}\left(\Gamma\sigma\Gamma\right)_+{}^- + \tilde{\kappa}_+{}^- + c_1\left(\Gamma_+{}^+ - \Gamma_-{}^- - z\right) + c_0\,\Gamma_+{}^0 + c_{21}\,,\\
\kappa_{22} &= -V_0\,\omega^0 - V_1\,\omega^1 - \Gamma_0{}^0 - \Gamma_+{}^+ - c_{00} - c_{11}\,,
\end{aligned} \tag{C.14}$$

where σ is a constant matrix which, in the (t,ϕ_+,ϕ_-) basis, is given by

$$\sigma = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1 \end{pmatrix}\,. \tag{C.15}$$

The c_{ij}'s are constants that are fixed by requiring that asymptotically, the κ matrix approaches

$$\kappa \approx -\frac{\delta\chi}{4}\,\cos 2\theta\, d\phi_- + O\left(\tfrac{1}{r^2}\right)\,. \tag{C.16}$$

For our solution (4.24), we can obtain the correct asymptotics for the κ matrix by fixing these constants as

$$\begin{aligned}
c_{00} &= \frac{R^2\lambda(1-\nu)}{2(1-\lambda)}\,,\\
c_{01} = c_{20} &= -\frac{R^3 C_1(1-\nu)\left[(1-\lambda+\bar{b}_4 C_2(1-\nu)(1+\nu-2\lambda\nu)\right]}{4(1-\lambda)\left[1-\lambda+\bar{b}_4 C_2(1+\lambda)(1-\nu)^2\right]}\,,\\
c_{02} = c_{10} &= \frac{R\,\bar{b}_4 C_1 C_2(1-\nu)^2}{1-\lambda+\bar{b}_4 C_2(1+\lambda)(1-\nu)^2}\,,\\
c_{11} &= -\frac{R^2\left[\bar{b}_4(\lambda-\nu)(1-\nu)^2 + C_2\,\nu(1-\lambda)\right]}{2(1-\lambda)\left[\bar{b}_4(1-\nu)^2 + C_2\right]}\,,\\
c_{12} &= 0\,,\\
c_{21} &= -\frac{R^4\lambda(\lambda-\nu)(1-\nu)}{8(1-\lambda)^2}\,.
\end{aligned} \tag{C.17}$$

Once this constants are fixed, the κ matrix is uniquely determined; the explicit expression is very long and we will not give it.

D Conserved charges

Upon reduction along the KK circle parametrized by the coordinate ξ^1 in (4.54) we obtain a four dimensional asymptotically flat solution. This solution consists in an KK electrically

charged rotating black hole separated from the nut, which accounts for the magnetic charge. We can easily compute the conserved charges of the four-dimensional solution, and we find:

$$M_{\rm tot} = \frac{\ell}{16\,G_4\,(1+b_4)(1-\nu^2)^2}\Bigg\{(1+\nu)^2\Big[D_1\,c_{2\gamma}(3+{\rm ch}_{2\alpha}) - 6\big(D_1 - 4\,\hat{R}^2\,\nu(1-\nu)(1+b_4)\big)\,{\rm sh}_\alpha^2 + 4\,D_3\,s_\gamma\,{\rm sh}_{2\alpha}\Big] + \hat{R}\,D_\nu\,\nu(1-\nu)\Big[\,D_2\,s_{2\gamma}(3+{\rm ch}_{2\alpha}) + 2\big(D_2+8\,b_4(1-\nu)\big)\,c_\gamma\,{\rm sh}_{2\alpha}\Big]\Bigg\}\,, \tag{D.1a}$$

$$Q = \frac{\ell}{8(1+b_4)(1-\nu^2)^2}\Bigg\{\Big[(1+\nu)^2 D_1\,c_{2\gamma} - 3(1+\nu)^2\big(D_1 - 4(1+b_4)\hat{R}^2\,\nu(1-\nu)\big) + \,\hat{R}\,D_\nu\,D_2\,\nu(1-\nu)\,s_{2\gamma}\Big]{\rm sh}_{2\alpha} + \Big[4\,(1+\nu)^2\,D_3\,s_\gamma + 2\,\hat{R}\,D_\nu\,\nu(1-\nu)\big(D_2+8\,b_4(1-\nu)\big)c_\gamma\Big]\,{\rm ch}_{2\alpha}\Bigg\}\,, \tag{D.1b}$$

$$P = \frac{\ell}{8|1+b_4|(1-\nu^2)^2}\Bigg\{2\Big[\hat{R}\,D_\nu\,D_2\,\nu(1-\nu)\,c_{2\gamma} - (1+\nu)^2 D_1\,s_{2\gamma}\Big]\,{\rm sh}_\alpha + \Big[-2\,\hat{R}\,D_\nu\,\nu(1-\nu)\big(D_2+8\,b_4(1-\nu)\big)\,s_\gamma + 4\,(1+\nu)^2 D_3\,c_\gamma\Big]\,{\rm ch}_\alpha\Bigg\}\,, \tag{D.1c}$$

$$J = \frac{R^2}{16\,G_4\,(1+b_4)^2(1-\nu)^3(1+\nu)^2}\Bigg\{\Big[D_7\,(1+\nu)\,s_{2\gamma} + 2\,\hat{R}\,D_\nu\,D_8\,(1-\nu)\,\nu\,c_{2\gamma}\Big]\,{\rm ch}_\alpha - 2\,\Big[\hat{R}\,D_\nu\,(D_8+D_9)(1-\nu)\,\nu\,s_\gamma + (D_7 + D_{10}\,\hat{R}^2\,\nu^2)(1+\nu)\,c_\gamma\Big]{\rm sh}_\alpha\Bigg\}\,, \tag{D.1d}$$

Similarly, the horizon area, temperature and angular velocity are found to be

$$\mathcal{A}_4 = \frac{\pi\,R^2\,\nu^2}{2(1+b_4)^2(1-\nu^2)^{5/2}}\Bigg\{\Big[2\,\hat{R}\,D_\nu\,D_4\,(1-\nu)(1+\nu+b_4(1-\nu)) - D_5\,(1+b_4)(1+\nu)^2 s_{2\gamma} - + 2\,\hat{R}\,D_\nu(1+b_4)(1-\nu^2)(D_4 - 2\,\nu\,\hat{R}^2(1+\nu+b_4(1-\nu)))c_{2\gamma}\Big]{\rm ch}_\alpha + \Big[(1+\nu)^2 D_6\,c_\gamma + 4\,\hat{R}\,D_\nu\,\nu(1-\nu)(D_2+8\,b_4(1-\nu))s_\gamma\Big]{\rm sh}_\alpha\Bigg\}\,, \tag{D.2a}$$

$$T_{\rm H} = \frac{2(1+b_4)^2(1-\nu^2)^{5/2}}{\ell\,\pi\,\nu}\Bigg\{\Big[2\,\hat{R}\,D_\nu\,D_4\,(1-\nu)(1+\nu+b_4(1-\nu)) - D_5\,(1+b_4)(1+\nu)^2 s_{2\gamma} - + 2\,\hat{R}\,D_\nu(1+b_4)(1-\nu^2)(D_4 - 2\,\nu\,\hat{R}^2(1+\nu+b_4(1-\nu)))c_{2\gamma}\Big]{\rm ch}_\alpha + \Big[(1+\nu)^2 D_6\,c_\gamma + 4\,\hat{R}\,D_\nu\,\nu(1-\nu)(D_2+8\,b_4(1-\nu))s_\gamma\Big]{\rm sh}_\alpha\Bigg\}^{-1}\,, \tag{D.2b}$$

$$\Omega_{\rm H} = \frac{8|1+b_4|(1-\nu^2)^2}{\ell}\Bigg\{\Big[2\,\hat{R}\,D_\nu\,D_4\,(1-\nu)(1+\nu+b_4(1-\nu)) - D_5\,(1+b_4)(1+\nu)^2 s_{2\gamma} -$$

$$+ 2\,\hat{R}\,D_\nu(1+b_4)(1-\nu^2)(D_4 - 2\,\nu\,\hat{R}^2(1+\nu+b_4(1-\nu)))c_{2\gamma}\Big]\mathrm{ch}_\alpha$$
$$+ \Big[(1+\nu)^2 D_6\,c_\gamma + 4\,\hat{R}\,D_\nu\,\nu(1-\nu)(D_2 + 8\,b_4(1-\nu))s_\gamma\Big]\mathrm{sh}_\alpha\Big\}^{-1}, \tag{D.2c}$$

To simplify the expressions for the various magnitudes in equations (D.1)-(D.2), we have defined $c_\gamma \equiv \cos\gamma$, $s_\gamma \equiv \sin\gamma$, and $\mathrm{ch}_\alpha \equiv \cosh\alpha$, $\mathrm{sh}_\alpha = \sinh\alpha$, which in turn are fixed according to (4.47) and (4.53) respectively. Similarly, the constants D_i are given by

$$D_\nu = (1+\nu)\sqrt{\frac{1+\nu}{1-\nu}}\,, \tag{D.3a}$$
$$D_1 = (1-b_4) - \big[2(1-b_4) - 3(1+b_4)\,\hat{R}^2\big]\,\nu + \big[(1-b_4) - 3\,(1+b_4)\hat{R}^2 - \tfrac{1}{2}(1+b_4)\hat{R}^4\big]\,\nu^2$$
$$- \tfrac{1}{4}(1-b_4)\,\hat{R}^4\,\nu^4\,, \tag{D.3b}$$
$$D_2 = \hat{R}^2(1+\nu)^2 - 4\,b_4\big[1-\nu-\tfrac{1}{4}\hat{R}^2(1+2\,\nu-\nu^2)\big]\,, \tag{D.3c}$$
$$D_3 = (1-b_4)\big[(1-\nu)^2 + \tfrac{1}{4}\hat{R}^4\,\nu^4\big] + \tfrac{1}{2}(1+b_4)\hat{R}^4\,\nu^2\,. \tag{D.3d}$$
$$D_4 = 4\,(1-\nu) + \hat{R}^2\,(1+b_4+(1-b_4)\,\nu^2) \tag{D.3e}$$
$$D_5 = 4\Big[(1-\nu)^2 - 3\,\hat{R}^2(1-\nu)\,\nu + \frac{1}{4}\hat{R}^4\,\nu^4\Big] - \frac{1}{2}(1+b_4)\,\hat{R}^4\,(1-\nu^2)^2 \tag{D.3f}$$
$$D_6 = (1+b_4)\Big\{8\Big[(1-\nu)^2 + \hat{R}^2\,(1-\nu) - \frac{1}{4}\hat{R}^4\,\nu^4\Big] + (1+b_4)\,\hat{R}^4\,(1-\nu^2)^2\Big\}$$
$$+ 8\,(1-b_4)\,\hat{R}^2\,(1-\nu)\,\nu^2 \tag{D.3g}$$
$$D_7 = \frac{1}{2}(1+\nu)\,[4\,(1-\nu)^2(1-\nu+2\nu^2) - 12\,\hat{R}^2\,(1-\nu^2)\,\nu^2 - \hat{R}^4\,(1-2\nu-\nu^3)\,\nu^3]$$
$$- b_4\hat{R}^4(1-\nu^2)^2\nu^3 - \frac{1}{2}b_4^2\,(1-\nu)\,[4\,(1-\nu)^2(1+\nu+2\nu^2) - 12\,\hat{R}^2\,(1+2\nu-3\nu^2)\,\nu^2$$
$$+ \hat{R}^4\,(1+2\nu+\nu^3)\,\nu^3] \tag{D.3h}$$
$$D_8 = 2\,\hat{R}^2\,(1+\nu^2)\,\nu^2 - 2\,(1-b_4^2)\,(1+\nu)^2\,(1-\nu) - 2\,b_4^2\,(1-\nu)\,\nu^2\,(4+\hat{R}^2(1+\nu))$$
$$- (1+b_4)^2\,\hat{R}^2\,(1-\nu)^2\,\nu^2 \tag{D.3i}$$
$$D_9 = 4\,(1-\nu)\,[(1+\nu)^2\,(1-b_4^2) + 4\,b_4^2\,\nu^2] \tag{D.3j}$$
$$D_{10} = (1+\nu)\,[6\,(1-\nu^2) + \hat{R}^2\,(1-2\nu-\nu^3)\,\nu] + 2\,b_4\,\hat{R}^2\,(1-\nu^2)^2\,\nu$$
$$- b_4^2\,(1-\nu)\,[6\,(1+2\nu-3\nu^2) - \hat{R}^2\,(1+2\nu+\nu^3)\,\nu] \tag{D.3k}$$

We have checked numerically that if the parameters ν and $\hat{R}$ are constraint to vary in the ranges (4.56), the mass M, the temperature T_{H} and the horizon area $\mathcal{A}_{\mathrm{H}}$ are always positive.

In terms of the four dimensional quantities, the five-dimensional angular momenta corresponding to the original angles (4.2) are given by

$$J^{(5)}_{\hat{\psi}} = \frac{P\,Q}{G_4} + J\,, \qquad J^{(5)}_{\hat{\phi}} = \frac{P\,Q}{G_4} - J\,. \tag{D.4}$$

Note that the angles $\hat{\psi}$ and $\hat{\phi}$ are related to the angles ψ and ϕ used in sec. 3 as

$$\hat{\psi} = \frac{\psi}{2}\,, \quad \hat{\phi} = \frac{\psi}{2} - \phi\,, \tag{D.5}$$

and thus the corresponding conserved charges are related as

$$J_\psi = \frac{1}{2}(J^{(5)}_{\hat{\psi}} + J^{(5)}_{\hat{\phi}}) = \frac{P\,Q}{G_4}\,, \quad J_\phi = -J^{(5)}_{\hat{\phi}} = J - \frac{P\,Q}{G_4}\,. \tag{D.6}$$

The horizon area of the five-dimensional solution is given by

$$\mathcal{A}_5 = 2\pi\,L\,\mathcal{A}_4\;, \tag{D.7}$$

where L is defined below (D.9). The temperature of the horizon of the four- and the five-dimensional solutions coincides. The mass of the five-dimensional solution can be computed as

$$M = M_{\rm tot} - M_{\rm D6}\;, \tag{D.8}$$

where $M_{\rm D6}$ is the mass of the KK monopole (6.2).

Finally we notice that regularity of the *new* five-dimensional metric (4.54) imposes that ξ^1 has to be periodically identified as

$$\xi^1 \sim \xi^1 + 2\pi L\;, \qquad L = \frac{4P}{N_6}\;, \tag{D.9}$$

for an integer N_6.

E Physical magnitudes of the approximate doubly spinning solution

It is very easy to include in our perturbative construction a second independent angular momentum along the S^2 of the ring. We need only use the stress tensor that reproduces the long-distance field of a boosted Kerr black string. This turns out to be very simple, since it takes the same form as (2.4), plus an additional component $T_{\tau\phi}$ for the spin along ϕ. Actually, we do not even need the details of this component, since it falls off sufficiently fast at infinity so as to not affect the equilibrium equations. To obtain the value of the spin J_ϕ of the ring, we need simply compute it for the boosted Kerr string at the equilibrium boost.

Then, the five-dimensional physical magnitudes of the approximate doubly spinning black ring in Taub-NUT (so $b=0$) at equilibrium (so $\sinh^2\alpha = 1$) are found to be

$$M = \frac{3}{4\,G_5}\,r_0\,\Delta z\;, \tag{E.1a}$$

$$J_\psi = \frac{\sqrt{2}}{16\pi\,G_5}\,r_0\,(\Delta z)^2\;, \qquad J_\phi = \frac{\sqrt{2}}{2G_5}\,r_0\,a\,\Delta z\;, \tag{E.1b}$$

$$\mathcal{A}_5 = 4\pi\sqrt{2}\,(r_+^2 + a^2)\Delta z\;, \tag{E.1c}$$

where $r_+ = 2r_0 + \sqrt{(2r_0)^2 - a^2}$ and a is the Kerr rotation parameter. The four-dimensional magnitudes are obtained using

$$Q = \frac{G_4 J_\psi}{P}\,, \qquad J = J_\psi + J_\phi\,, \tag{E.2}$$

as follows from (3.19), (3.20), and (3.24). The four-dimensional area is obtained as in (D.7). Clearly, $G_4 J \neq QP$ if $a \neq 0$.

For the exact solution, the configuration in which the ring is far from the NUT is described by the limit in which $\nu \to 0$ and $\hat{R} = \sqrt{2}(1 - \eta\nu)$, for some fixed η. In this limit the mass, charges and area of the exact solution become

$$\begin{aligned}
\frac{G_4 M_{\rm tot}}{P} &= \frac{3+\eta}{2\eta} - \frac{18\eta^2 - 16\eta + 3}{8\eta^2}\nu\,, \\
\frac{Q}{P} &= \frac{\sqrt{2}}{\eta} - \frac{10\eta^2 - 12\eta + 1}{2\sqrt{2}\eta^2}\nu\,, \\
\frac{G_4 J}{P^2} &= \frac{\sqrt{2}}{\eta} - \frac{10\eta^2 - 12\eta - 3}{2\sqrt{2}\eta^2}\nu\,, \\
\frac{\mathcal{A}_4}{P^2} &= \frac{16\sqrt{2}\pi}{\eta^2} - 8\sqrt{2}\pi\frac{4\eta^2 - 2\eta + 1}{\eta^3}\nu\,.
\end{aligned} \tag{E.3}$$

It can be checked that these values match the ones of the perturbative solution given in (E.1) after the following reparametrization:

$$\Delta z = 2\pi L\Big(1 - \frac{3\eta - 5}{3}\nu\Big)\,, \quad r_0 = \frac{L}{2\eta}\Big(1 - \frac{6\eta^2 + 4\eta + 3}{12\eta}\nu\Big)\,, \quad a = \frac{\nu}{2\eta}\,. \tag{E.4}$$

One can also check that the exact metric reduces, at first order in ν and after some change of coordinates, to that of a boosted Kerr black string, with the angular momentum parameter a given in (E.4) and boost parameter given by $\sinh\alpha = 1 + 2\nu$.

References

[1] R. Emparan and H. S. Reall, "Black Holes in Higher Dimensions," Living Rev. Rel. **11** (2008) 6 [arXiv:0801.3471 [hep-th]].

[2] N. A. Obers, "Black Holes in Higher-Dimensional Gravity," arXiv:0802.0519 [hep-th].

[3] V. Niarchos, "Phases of Higher Dimensional Black Holes," Mod. Phys. Lett. A **23**, 2625 (2008) [arXiv:0808.2776 [hep-th]].

[4] R. Emparan and H. S. Reall, "Generalized Weyl solutions," Phys. Rev. D **65** (2002) 084025 [arXiv:hep-th/0110258].

[5] R. Emparan and H. S. Reall, "A rotating black ring in five dimensions," Phys. Rev. Lett. **88** (2002) 101101 [arXiv:hep-th/0110260].

[6] R. Emparan and H. S. Reall, "Black rings," Class. Quant. Grav. **23** (2006) R169 [arXiv:hep-th/0608012].

[7] A. A. Pomeransky and R. A. Sen'kov, "Black ring with two angular momenta," arXiv:hep-th/0612005.

[8] H. Elvang and P. Figueras, "Black Saturn," JHEP **0705** (2007) 050 [arXiv:hep-th/0701035].

[9] H. Iguchi and T. Mishima, "Black di-ring and infinite nonuniqueness," Phys. Rev. D **75** (2007) 064018 [arXiv:hep-th/0701043].

[10] S. Giusto and A. Saxena, "Stationary axisymmetric solutions of five dimensional gravity," Class. Quant. Grav. **24** (2007) 4269 [arXiv:0705.4484 [hep-th]].

[11] J. Evslin and C. Krishnan, "The Black Di-Ring: An Inverse Scattering Construction," arXiv:0706.1231 [hep-th].

[12] J. Ford, S. Giusto, A. Peet and A. Saxena, "Reduction without reduction: Adding KK-monopoles to five dimensional stationary axisymmetric solutions," arXiv:0708.3823 [hep-th].

[13] H. Elvang and M. J. Rodriguez, "Bicycling Black Rings," JHEP **0804** (2008) 045 [arXiv:0712.2425 [hep-th]].

[14] R. Emparan, T. Harmark, V. Niarchos, N. A. Obers and M. J. Rodriguez, "The Phase Structure of Higher-Dimensional Black Rings and Black Holes," JHEP **0710** (2007) 110 [arXiv:0708.2181 [hep-th]].

[15] M. M. Caldarelli, R. Emparan and M. J. Rodriguez, "Black Rings in (Anti)-deSitter space," JHEP **0811** (2008) 011 [arXiv:0806.1954 [hep-th]].

[16] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, "Supersymmetric 4D rotating black holes from 5D black rings," JHEP **0508** (2005) 042 [arXiv:hep-th/0504125].

[17] I. Bena, P. Kraus and N. P. Warner, "Black rings in Taub-NUT," Phys. Rev. D **72** (2005) 084019 [arXiv:hep-th/0504142].

[18] D. Gaiotto, A. Strominger and X. Yin, "5D black rings and 4D black holes," JHEP **0602** (2006) 023 [arXiv:hep-th/0504126].

[19] W. Taylor, "Adhering 0-branes to 6-branes and 8-branes," Nucl. Phys. B **508**, 122 (1997) [arXiv:hep-th/9705116].

[20] D. Rasheed, "The Rotating dyonic black holes of Kaluza-Klein theory," Nucl. Phys. B **454** (1995) 379 [arXiv:hep-th/9505038].

[21] F. Larsen, "Rotating Kaluza-Klein black holes," Nucl. Phys. B **575** (2000) 211 [arXiv:hep-th/9909102].

[22] R. Emparan and G. T. Horowitz, "Microstates of a neutral black hole in M theory," Phys. Rev. Lett. **97** (2006) 141601 [arXiv:hep-th/0607023].

[23] R. Emparan and A. Maccarrone, "Statistical description of rotating Kaluza-Klein black holes," Phys. Rev. D **75**, 084006 (2007) [arXiv:hep-th/0701150].

[24] B. Chen, H. Itoyama, T. Matsuo and K. Murakami, "p p' system with B field, branes at angles and noncommutative geometry," Nucl. Phys. B **576** (2000) 177 [arXiv:hep-th/9910263].

[25] M. Mihailescu, I. Y. Park and T. A. Tran, "D-branes as solitons of an N = 1, D = 10 non-commutative gauge theory," Phys. Rev. D **64** (2001) 046006 [arXiv:hep-th/0011079].

[26] E. Witten, "BPS bound states of D0-D6 and D0-D8 systems in a B-field," JHEP **0204**, 012 (2002) [arXiv:hep-th/0012054].

[27] Ki-Myeong Lee, Eoin Ó Colgáin, Hossein Yavartanoo and K. P. Yogendran "BPS D0-D6 Branes in Supergravity," [arXiv:0811.2086 [hep-th]].

[28] B. Carter, "Essentials of classical brane dynamics," Int. J. Theor. Phys. **40**, 2099 (2001) [arXiv:gr-qc/0012036].

[29] H. Elvang, R. Emparan and A. Virmani, "Dynamics and stability of black rings," JHEP **0612** (2006) 074 [arXiv:hep-th/0608076].

[30] S. Giusto, S. D. Mathur and Y. K. Srivastava, "A microstate for the 3-charge black ring," Nucl. Phys. B **763**, 60 (2007) [arXiv:hep-th/0601193].

[31] B. Bates and F. Denef, "Exact solutions for supersymmetric stationary black hole composites," arXiv:hep-th/0304094.

[32] J. B. Gutowski, D. Martelli and H. S. Reall, "All supersymmetric solutions of minimal supergravity in six dimensions," Class. Quant. Grav. **20** (2003) 5049 [arXiv:hep-th/0306235].

[33] D. Maison, "Ehlers-Harrison Type Transformations For Jordan's Extended Theory Of Gravitation," Gen. Rel. Grav. **10**, 717 (1979).

[34] T. Harmark, "Stationary and axisymmetric solutions of higher-dimensional general relativity," Phys. Rev. D **70** (2004) 124002 [arXiv:hep-th/0408141].

[35] V. A. Belinsky and V. E. Zakharov, "Integration Of The Einstein Equations By The Inverse Scattering Problem Technique And The Calculation Of The Exact Soliton Solutions," Sov. Phys. JETP **48** (1978) 985 [Zh. Eksp. Teor. Fiz. **75** (1978) 1953].

[36] V. Belinski and E. Verdaguer, "Gravitational solitons," Cambridge, UK: Univ. Pr. (2001) 258 p.

[37] R. Emparan, "Rotating circular strings, and infinite non-uniqueness of black rings," JHEP **0403** (2004) 064 [arXiv:hep-th/0402149].

[38] Y. Chen and E. Teo, "A rotating black lens solution in five dimensions," Phys. Rev. D **78** (2008) 064062 [arXiv:0808.0587 [gr-qc]].

[39] M. S. Costa and M. J. Perry, "Interacting black holes," Nucl. Phys. B **591** (2000) 469 [arXiv:hep-th/0008106].

[40] T. Regge, "General Relativity without coordinates," Nuovo Cim. **19**, 558 (1961).

[41] J. J. Blanco-Pillado, R. Emparan and A. Iglesias, "Fundamental Plasmid Strings and Black Rings," JHEP **0801**, 014 (2008) [arXiv:0712.0611 [hep-th]].